Chain-Consistent Tight Bounds of True Index Numbers of Productivity: An Application to EU KLEMS Data

Working paper nr. 34

Milana, Carlo

EU KLEMS WORKING PAPER SERIES Product

Chain-Consistent Tight Bounds of True Index Numbers of Productivity: An Application to EU KLEMS Data

Working paper nr. 34

Milana, Carlo

EU KLEMS Project
Productivity in the European Union: A Comparative Industry Approach

August 2008

This project is funded by the European Commission, Research Directorate General as part of the 6th Framework Programme, Priority 8, "Policy Support and Anticipating Scientific and Technological Needs".

Chain-Consistent Tight Bounds of True Index Numbers of Productivity: An Application to EU KLEMS Data

By Carlo Milana°

Abstract

It has been seldom recognized, after an early clarification made by Samuelson (1947, p. 151) and successive systematic developments of Afriat, that chain-consistent relative price levels can be reconstructed (up to a constant) using several observation points directly and simultaneously. The derived ratios of price levels are mutually consistent by construction and thereby satisfy the circularity or transitivity test as well as other desirable requirements. By contrast, to the best of our knowledge, all the other existing index number methods fall into the realm of the so-called "impossibility theorem", failing in particular the circularity test intrinsically. This paper builds on the recent reappraisal of this approach by Afriat and Milana (2008) and compares it with akin non-parametric techniques that are based on revealed preferences. An application to productivity measurement using EU KLEMS data highlights the usefulness of the method.

Key words: Aggregation, Index number theory Non-parametric analysis, Price level, Price index, Productivity measurement.

1. Introduction

The aim of this paper is to construct and apply empirically chain consistent tight upper and lower bounds of "true" measures of productivity in a multilateral setting. The analysis starts from the recognition that any index number formula may be potentially unsuitable as an approximation of the unknown "true" index, if this exists at all. We can, however, identify the tight upper and lower bounds, if any, and test simultaneously the consistency of the data with aggregation conditions.

The approach presented here is derived directly from that defined by Afriat (2005) and applied by Afriat and Milana (2008). It is much simpler and appears to be more convenient than the usual non-parametric deterministic methods based on the

---

° Istituto di Studi e Analisi Economica, Rome, and Department of Management, Birkbeck University of London, e-mail address: carlo.milana@iol.it. This paper, has been prepared for the EU KLEMS project funded by the European Commission, Research Directorate General as part of the 6th Framework Programme and for presentation at the 2008 World Congress on National Accounts and Economic Performance Measures for Nations, Key Bridge Marriott, Washington D.C., May 13-17, 2008. The views expressed in this paper are not necessarily those of the institutions and persons mentioned here.

"revealed preference" theory originally associated with Samuelson (1938a)(1938b)(1938c)(1947)(1948) and Afriat (1967b)(1972)(1977) himself and with the further clarifications of Diewert (1973) and Varian (1982)(1983)(1984). Most of these rely on *effective* observed budget lines rather than (observed and virtual) budget lines passing through the *base* observation points. The consistency with aggregation conditions has appeared to be violated rarely in the literature even in the domain of consumer behavior. These unexpected results have puzzled by many authors, who have questioned the power of the test itself.

It is believed that these results have been obtained because income effects, which tend to dominate the level of demand, overcome the price-induced substitution effects (see, for example, Varian, 1983, 2006, Blundell *et al.* 2003, and Blundell, 2005). The budget lines taken into exam seldom cross each other and, for this reason, there would be little room for the violation of the "revealed preference" conditions, which is expected in the general non-homothetic case.

The problem may be avoided by considering that, in the traditional economic model, the relevant isoquants or indifference curves, if any, are bounded not by the *effective* budget lines, but by piecewise linear frontiers passing through the same *base point* and representing, at different observed prices, the constant-quantity monetary income. The algorithm proposed by Afriat offers a way to construct tight upper and lower bounds of the unknown "true" economic index numbers, if the aggregation conditions are satisfied. The two piecewise linear bounds may themselves be good candidates of representing the "true" index number. From the point of view of this approach, the question itself of the power of non-parametric tests raised by some authors arises only because of the nature of the measures used.

In case of inconsistency with aggregation conditions, it is suggested that the accounting system should be widened in order to be consistent with a more general set of the variables (by considering, for example, a more complete set of non-separable outputs and inputs) and, if this is not possible, a correction for allocative inefficiency is introduced. The proposed method is truly constructive in that it does not even necessarily require a certain model or the existence of non-observational objects like utility- or technology-based functions (or the corresponding "dual" value functions such as optimal cost, revenue, or profit functions).

The alternative index numbers obtained as tight bounds of the unknown "true" economic index are derived from aggregates of *price levels*. Since these are by nature transitive in their ratios or differences, the index numbers that are derived in this way in a multilateral setting are *chain-consistent* and therefore satisfy automatically the requirement of transitivity (or circularity) property. To our knowledge, this approach is, until now, the only one available in the field of index numbers that satisfies this requirement directly without further manipulations. The proposed methodology is applied to the EUKLEMS database, which has been built for the European Commission covering more than 30 countries at the industry level since 1970.

The paper is organized as follows. Section 2 briefly recalls the problems encountered with multilateral index numbers. Section 3 reviews the Samuelson-Afriat's non-parametric (deterministic) analysis based on revealed preference and the alternative index number approach presented here. Section 4 describes Afriat's power algorithm and the program performing the empirical computations. Section 5 formulates the problem of multifactor productivity measurement. Section 6 describes some of the results obtained in productivity measurement using EU KLEMS data. Section 7 concludes.

2. Aggregation problems with index numbers

Prices (or deflators) and quantities (or volumes) referring to economic aggregates are not observable. The empirical literature has followed alternative directions to construct such aggregates, among which the most common are: (*i*) multiple observations are introduced simultaneously and analyzed using the Samuelson-Afriat revealed preference techniques; (*ii*) assuming specific functional forms about the underlying economic functions (cost or utility or production functions), the corresponding "exact" index numbers are applied (using the terminology of Byushgens, 1924 and Konüs and Byushgens, 1926); (*iii*) the aggregator functions are estimated by means of econometric techniques and their numerical values are assessed by choosing particular levels of reference variables. All these approaches present serious drawbacks.

Revealed preference techniques are primarily used to test the compatibility of the data with the rational behaviour hypothesis as well as for linear homogeneity and separability restrictions, which are necessary and sufficient for the existence of an economic aggregator function. In the empirical literature, these analytical techniques routinely fail to reject homotheticity, thus suggesting that there is something wrong with the assumptions on the behaviour generating the data and/or with the method itself, which appears not to be demanding enough for various reasons. The power of the so-called Afriat-Varian test is considered too low so that some authors have tried to extend it in order to take account of non-homothetic changes.

The economic index number approach assumes an underlying specific well-behaved functional form of the aggregator function and identifies the corresponding exact index number. This approach gives rise to a joint test of the specific functional form and the fundamental requirements for the measures obtained. Given the assumptions made, the failure to satisfy the requirements cannot be attributable to the data or the model taken separately from the particular specification of the functional form. Moreover, the interpretation of the index numbers formulas as being exact for aggregator functions providing approximations of the "true" measure presents its own limits, as pointed out by a number of analysts. This applies also to the class of the "superlative" index numbers that are supposed to provide approximations up to the second order (see, for example, Afriat, 1977, Uebe, 1978, Neary, 2004, Milana, 2005, Hill, 2006a, Afriat and Milana, 2008).

The economic approach assumes that the prices represented here with the vector *p* and the respective quantities *q* of demanded goods are consistent with maximization of utility, which is a well-behaved (concave) function of the quantities *q*, let us say $\phi(q)$, governing the demand subject to the budget constraint given by total income $E = pq = \sum_i p_i q_i$. This could also be seen as minimization of expenditure subject to a given utility, that is

$$E(p,u) = Min_q \{pq : \phi(q) \geq u\}, \tag{1}$$

where *u* is the given utility.

Following Samuelson and Swamy (1974, p. 570), for a given value function $E(p,u)$, meaningful aggregates of *p* and *q* can be constructed using the following results:

Rule (*i*): $q = \nabla_p E(p,u)$ by *Shephard's lemma* (established under the hypothesis of optimizing behaviour), which describes the demand of *q* as functions of prices and utility, governed by preferences represented by the function $f(q)$ under the constraint of disposable income whose value is equal to pq.

Rule (*ii*): $E(p,u) = e(p) \cdot f(q)$ by *Shephard-Afriat's factorization theorem* (established under the hypothesis of *homotheticity* of the utility function $f(q)$). This rule defines the invariancy of the aggregator functions *e*(*p*) and *f*(*q*) with respect to the reference variables *u* and *p* respectively[1]. As Samuelson and Swamy (1974, p. 570) have recognized, "[t]he invariance of the price index is seen to imply and to be implied by the invariance of the quantity index from its reference price base".

Under the rules (*i*) and (*ii*), economic index numbers P_{t0} and Q_{t0} can be constructed, which are called "exact" for the aggregator functions $e(p)$ and $f(q)$, respectively, as they are identically equal to the ratios of the values taken by the aggregates at two observed points, that is $P_{t0} = \dfrac{e(p_t)}{e(p_0)}$ and $Q_{t0} = \dfrac{f(q_t)}{f(q_0)}$. As noted by Samuelson and Swamy (1974, p. 573, fn. 9) and Diewert (1976, pp. 132-133), all Fisher's tests are satisfied by all economic index number formulas (when these are

---

[1] Homotheticity and identical preferences seem, however, to have been noticed as early as the work of Antonelli (1886) as necessary and sufficient conditions for aggregation if this is to hold globally. Conditions for aggregation holding only locally and allowing preference eterogeneity have been studied by Afriat (1953-56) (1959) and Gorman (1953, 1961).

“exact” for the true aggregator functions in the homothetic case). Also the circularity test is satisfied, that is $P_{20} = P_{21} \cdot P_{10}$

as

$$\frac{e(p_2)}{e(p_0)} = \frac{e(p_1)}{e(p_0)} \cdot \frac{e(p_2)}{e(p_1)} \quad \text{with} \quad \frac{e(p_i)}{e(p_j)} = P_{ij} \qquad (2)$$

In principle, Byushgens (1924) and Konüs and Byushgens (1926) have introduced the concept of “exact” index numbers for the true aggregator function by showing that the popular index number formulas such as the Fisher “ideal” index number may yield the same numerical values of a specific aggregator function. Following Diewert (1976), let us assume that the aggregator function $e(p)$ has, for example, a *quadratic mean-of-order-r* functional form, that is $e_{Q^r}(p) \equiv (p^{r/2} A p^{r/2})^{1/r}$, where $-\infty \le r < 0,\ 0 < r \le \infty$, with A being a symmetric matrix whose positive elements a_{ij} satisfy the restriction $\iota' A \iota = 1$, where $\iota \equiv [11 \ldots 1]$, so that $e_{Q^r}(p) = 1$ if $p = \iota$.

We recall that McCarthy (1967) and Kadiyala (1972) considered the functional form $e_{Q^r}(p)$ as a generalization of a CES functional form, to which it is equivalent if all $a_{ij} = 0$ for $i \ne j$[2].

We have in fact

$$\frac{e_{Q^r}(p_1)}{e_{Q^r}(p_0)} = \frac{\left(p_1^{r/2} A p_1^{r/2}\right)^{1/r}}{\left(p_0^{r/2} A p_0^{r/2}\right)^{1/r}}$$

$$= \left[\frac{p_1^{r/2} A p_1^{r/2}}{p_0^{r/2} A p_1^{r/2}} \cdot \frac{p_1^{r/2} A p_0^{r/2}}{p_0^{r/2} A p_0^{r/2}} \right]^{1/r} \quad \text{since} \quad p_1^{r/2} A p_0^{r/2} = p_0^{r/2} A p_1^{r/2} \ \text{with a symmetric } A$$

$$= \left[\frac{p_1^{r/2} A p_1^{r/2}}{p_0^{r/2}\, \hat{p}_1^{1-r/2}\, \hat{p}_1^{r/2-1} A p_1^{r/2}} \cdot \frac{p_1^{r/2}\, \hat{p}_0^{1-r/2}\, \hat{p}_0^{r/2-1} A p_0^{r/2}}{p_0^{r/2} A p_0^{r/2}} \right]^{1/r} \quad \text{where } \wedge \text{ denotes a diagonal matrix}$$

formed with the elements of a vector

$$= P_{Q^r}(p_0, p_1, q_0, q_1) \equiv \left[\frac{\sum_i \frac{p_{1i}^{r/2}}{p_{0i}^{r/2}} s_{0i}}{\sum_i \frac{p_0^{r/2}}{p_1^{r/2}} s_{1i}} \right]^{\frac{1}{r}} \qquad (3)$$

---

[2] Moreover, Denny (1972)(1974) noted that, if $r = 1$, then $e_{Q^r}(p) \equiv (p^{r/2} A p^{r/2})^{1/r}$ reduces to the Generalized Leontief functional form proposed by Diewert (1969)(1971). Diewert (1976, p. 130) himself noted that, if $r = 2$, then it reduces to the Konüs-Byushgens (1926) functional form.

where $s_{ti} \equiv \frac{p_{ti}q_{ti}}{\sum_j p_{tj}q_{tj}} = \frac{p_{ti}^{r/2}\sum_j a_{ij}\, p_{tj}^{r/2}}{p_t^{r/2} A p_t^{r/2}}$ (with a_{ij} being the (*i*,*j*) element of matrix A), which is the observed value share of the *i*th quantity $q_{ti} = \partial E(p,u)/\partial p_{ti} = \frac{\partial e_{Q^r}(p)}{\partial p_{ti}} \cdot f(q) = \frac{p_{ti}^{\frac{r}{2}-1}\sum_j a_{ij}\, p_{tj}^{r/2}}{(p_t^{r/2} A p_t^{r/2})^{\frac{1}{r}-1}} \cdot f(q)$ by Shephard's lemma. The resulting formula $P_{Q^r}(p_0, p_1, q_0, q_1)$ is Diewert's (1976, p.131) *quadratic mean-of-order-r* index number,

The quadratic mean-of-order-*r* index number $P_{Q^r}(p_0, p_1, q_0, q_1)$ encompasses the class of all the index numbers that Diewert (1976) has called "superlative", which are exact for polynomial aggregator functions of degree up to the second order[3]. However, as already noted by Afriat (1956)(1967a) in the case of Fisher "ideal" index and Uebe (1978), starting from the index $P_{Q^r}(p_0, p_1, q_0, q_1)$, we cannot fully recover a polynomial of second order (with non-zero second derivatives) to represent the underlying aggregator function $e_{Q^r}(p)$. The index $P_{Q^r}(p_0, p_1, q_0, q_1)$ is constructed using the information of 2*N*-2 independent data concerning the weights s_0 and s_1, with N representing the number of items to be aggregated. By contrast, the underlying function $e_{Q^r}(p)$ has *N*(*N*+1)/2-1 independent elements in the symmetric matrix *A*, plus the power exponent *r*. With this last parameter exogenously given (assumed a priori), the free parameters outnumber the independent data concerning the weights when $N > 2$. Therefore, the numerical value of the index $P_{Q^r}(p_0, p_1, q_0, q_1)$ is arbitrarily determined by the choice of the value of *r* even with the minimum number of elements is set equal to 2. Moreover, solving the differential equations derived from the function $e_{Q^r}(p)$ for the elements of matrix *A* yields:

$$a_{ij} = \frac{e_{ij} - \frac{1-r}{e} e_i e_j}{\frac{r}{2} e^{1-r} (p_i p_j)^{r/2-1}} \qquad \text{for } i, j = 1, 2, \ldots, N \tag{4}$$

where $e = e_{Q^r}(p)$, $e_k = \frac{\partial e_{Q^r}(p)}{\partial p_k}$ and $e_{ij} \equiv \frac{\partial^2 e_{Q^r}(p)}{\partial p_i \partial p_j}$. Since non-zero elements a_{ij} are possible even if $C_{ij} = 0$, all the elements of the matrix *A* can be determined without

---

[3] Diewert (1976, p. 135) noted that, if $r = 1$, $P_{Q^r}(p_0, p_1, q_0, q_1)$ reduces to the implicit Walsh (1901, p. 105) index number (this index is exact for the Generalized Leontief aggregator function), and, if $r = 2$, then it reduces to the Fisher (1922) "ideal" index number (the geometric mean of the Laspeyres and Paasche indexes), which is exact for the Konüs-Byushgens (1926) aggregator function.

the need to have non-zero second-order derivatives. This implies that the same index $P_{Q^r}(p_0, p_1, q_0, q_1)$ for a given price and quantity data set could be equally "exact" for first-order and second-order polynomial aggregator functions. In other words, the superlativeness of the index $P_{Q^r}(p_0, p_1, q_0, q_1)$ is not guaranteed even in the homothetic case.

In an unpublished memorandum, Lau (1973) showed that, at the limit as r tends to zero, the functional form $e_{Q^r}(p)$ reduces to the homogeneous *translog aggregator function* so that

$$\lim_{r\to 0} e_{Q^r}(p) = e_T(p) \equiv \exp[\alpha_0 + a \cdot \ln p + \frac{1}{2} \ln p' \cdot A \cdot \ln p] \qquad (5)$$

Lau's proof is reported in Diewert (1980, p. 451) (see Milana, 2005 for an alternative proof).

Then

$$\lim_{r\to 0} \frac{e_{Q^r}(p_1)}{e_{Q^r}(p_0)} = \exp[\ln e_T(p_1) - \ln e_T(p_0)] \equiv \exp[\alpha_0 + a \cdot \ln p_1 + \frac{1}{2} \ln p_1^{'} \cdot A \cdot \ln p_1^{'}$$

$$-\alpha_0 - a \cdot \ln p_0 - \frac{1}{2} \ln p_0 ' \cdot A \cdot \ln p_0]$$

$$= \exp[(a + \ln p_0 A)(\ln p_1 - \ln p_0) + \frac{1}{2}(\ln p_1 - \ln p_0) A (\ln p_1 - \ln p_0)]$$

$$= \exp[(a + \ln p_1 A)(\ln p_1 - \ln p_0) - \frac{1}{2}(\ln p_1 - \ln p_0) A (\ln p_1 - \ln p_0)]$$

$$= \exp[\frac{1}{2}[(a + \ln p_0 A) + (a + \ln p_1 A)](\ln p_1 - \ln p_0)] \text{ taking the geometric average of the previous two lines}$$

$$= \exp[\frac{1}{2}[\nabla_{\ln p} e_T(p_0) + \nabla_{\ln p} e_T(p_0)](\ln p_1 - \ln p_0)]$$

$$= P_T(p_0, p_1, q_0, q_1,) \equiv \exp[\frac{1}{2}[s_0 + s_1](\ln p_1 - \ln p_0)], \text{ which is the the Törnqvist index,}$$

since $\partial \ln e_T(p) / \partial \ln p_i = \dfrac{\partial e_T(p)}{\partial p_i} \cdot \dfrac{p_i}{e_T(p)} = \dfrac{\partial e_T(p)}{\partial p_i} \cdot f(q) \cdot \dfrac{p_i}{e_T(p) \cdot f(q)} = s_i \equiv q_i \dfrac{p_i}{\sum_i p_i q_i}$ by Shephard's lemma, with $e_T(p) \cdot f(q) = \sum_i p_i q_i$ by Shephard-Afriat's factorization theorem.

An identification problem can be met with the true aggregator as a second-order polinomial function can be met starting also from the index number $P_T(p_0, p_1, q_0, q_1,)$. Following Uebe (1978), we can note that the superlativeness character of this index number cannot be present even in the homothetic case.

In empirical applications the above superlative index numbers generally fail to pass the circularity test. Fisher (1922) himself had noted that its “ideal” index number does not generally satisfy this test. Yet, as we have recalled above, if it is consistent with the data that are generated by an optimized behaviour governed by a homothetic utility and an expenditure function having a quadratic mean-of-order-2 functional form, at least locally, so that $P_{Q^2}(p_0, p_1, q_0, q_1) = P_{10}$, then also this index number should exhibit the transitivity property. Samuelson and Swamy (1974, p. 575) wittingly observe: “Where most of the older writers balk, however, is at the circular test that frees us from one base year. Indeed, so enamoured did Fisher become with his so-called Ideal index $[(p_1 q_0 / p_0 / q_0)(p_1 q_1 / p_0 / q_1)]^{1/2} =$ square root of (Laspeyres × Paasche) that, when he discovered it failed the circular test, he had the hubris to declare ‘…, therefore, a *perfect* fulfillment of this so-called circular test should really be taken as proof that the formula which fulfills it is erroneous’ (1922, p. 271). Alas, Homer has nodded; or, more accurately, a great scholar has been detoured on a trip whose purpose was obscure from the beginning”.

The circularity test may be violated either because the economic agent is not optimizing and/or the utility or technology function is non-homothetic, non-separable in the variables of interest and/or the chosen index number formula is not “exact” for the “true” aggregator function. This fact has been very seldom recognized in the economic literature even long after the notion of the exact index numbers has been discovered by Byushgens (1924) and Konüs and Byushgens (1926). This has prevented a widespread use of the circularity test as an empirical refutation of the aggregation conditions. We must emphasize the importance of this test as a first step to address the problem of aggregation correctly.

Violation of the circularity test may lead to severe problems of inconsistency not only in absolute levels of quantity indexes, but also in their relative ranking position. In a multilateral context, various methods have been devised so far to eliminate the inconsistencies of the results obtained. Most of them remain bilateral in nature, disguised in a multilateral dressing. Among these, we may mention the so-called star system where each observed point is compared with one observed or a hypothetical average point as in the so-called EKS and CDD methods, based respectively on the use of bilateral Fisher “ideal” and Törnqvist indexes. In these last two methods, the hypothetical average point used for an intermediate comparison is implicitly constructed as an average of all points. The main problem with these methods is the same as that encountered with bilateral index numbers that are not well grounded on the aggregation conditions. In the non-homothetic case, the conditions for the Shephard-Afriat’s factorization theorem do not hold. Therefore, the aggregator functions of prices and quantities are not well defined and, if we insist in constructing bilateral (economic) index numbers based on value functions (expenditure, profit or revenue functions), we will obtain spurious magnitudes of price and quantity aggregates being functions not only of individual prices and quantities respectively, but also of reference variables. This can be seen, for example, by inspecting the equations of the value shares *s* used as weights in the index number formulas considered above. In the non-homothetic case, the expenditure function cannot be decomposed into distinct terms of prices and utility. The quantities are obtained as follows

$$q_{ti} = \partial E(p,u)/\partial p_{ti} \quad \text{by Shephard's lemma} \tag{6}$$

with the shares given by

$$s_{it} = \frac{\partial E(p_t,u_t)}{\partial p_{ti}} \cdot \frac{p_{ti}}{E(p_t,u_t)} \tag{7}$$

which are functions not only of prices but also of the reference utility. They should be contrasted with the homothetic case where they are functions only of prices so that the aggregator function exists as a pure price component. A clear separation of price and quantity components of the total value changes is possible only in the special case of homothetic separability[4]. These conclusions differ from those of Caves, Christensen and Diewert (1982a)(1982b) and Diewert and Morrison (1986), where this distinction is not made[5].

When homothetic separability conditions are not met, any attempt to obtain deflated values by constructing linearly homogeneous quantity index numbers (using for example distance functions) would inevitably lead us to failure in satisfying the *exhaustiveness* requirement, which is better known in the literature as *(weak) factor reversal* test: the total value should be identically equal to the total contribution of the aggregate components. Our problem is to decompose the relative or absolute change in the scalar value $p_t \cdot q_t \equiv \sum p_{ti} q_{ti}$ between t=0 and t=1 into a scalar price-change component and a scalar quantity-change component starting from the change in the single elements of the two vectors p_t and q_t, that is

$$\sum p_{1i} q_{1i} - \sum p_{0i} q_{0i} = \Delta_P(p_0, p_1, q_0, q_1) + \Delta_Q(q_0, q_1, p_0, p_1) \tag{8}$$

---

[4] This is related to the well-known conclusion reached by Hulten (1973) on path-independency of Divisia indexes in the case of homothetic functions (see Milana, 1993 for further explanatory treatment).

[5] In the non-homothetic case, we might attempt to construct an aggregating linearly homogeneous quantity index numbers (price index numbers are always linearly homogeneous by construction). This is a procedure followed, for example, by Caves et. al. (1982) in defining their input quantity and price index numbers. However, the indexes thus obtained are not pure quantity and price components.

where $\Delta_P(p_0, p_1, q_0, q_1)$ and $\Delta_Q(q_0, q_1, p_0, p_1)$ are the additive price and quantity components of the absolute change in the scalar value *pq*, and

$$\frac{\sum p_{1i}q_{1i}}{\sum p_{0i}q_{0i}} = P(p_0, p_1, q_0, q_1) \cdot Q(q_0, q_1, p_0, p_1) \qquad (9)$$

where $P(p_0, p_1, q_0, q_1)$ and $Q(q_0, q_1, p_0, p_1)$ are the moltiplicative price and quantity components of the relative change in the scalar value *pq*. These price and quantity components should be appropriate *pure* aggregates of the changes in the elementary prices and elementary quantities, respectively: *each of these two aggregating components should not contain elements of the other*. If changes in relative quantities are affected only by changes in relative prices, as it happens in the homothetic case, then the change in the scalar value $\sum p_i^t q_i^t$ can be split, at least in theory, in two price and quantity components, representing proportional (scale) price and quantity factors, respectively. By contrast, if changes in quantities are affected not only by relative prices, but also by some "external" or "reference" variable in a non-homothetic way, then it is impossible to disentangle completely the effects on quantities arising from the changes in relative prices and the changes in the reference variable. Any definition and measure of the price and quantity change components would result to be spurious magnitudes, for each component contains some elements of the other. Under this condition, the circularity test cannot be generally satisfied.

Pollak (1971), Samuelson and Swamy (1974, 576-77), Archibald (1977), Fisher (1988)(1995) and Fisher and Shell (1998) had observed that, in the general non-homothetic case, a side effect of the non-invariancy of the price aggregating index is that the corresponding quantity index fails to satisfy the requirements of the linear homogeneity test. For example, if all the single quantities double, the derived quantity measure fails to double[6]. This undesired property is a consequence of the fact that, in the non-homothetic case, the price aggregating index is a spurious index number which captures not only the changes in prices but also changes in the reference utility level.

In a later work, Diewert (1983a, pp. 178-179) has recognized that, in the non-homothetic case, a quantity index number obtained implicitly by deflating the index of total nominal expenditure by means of an economic price index may not result to be linearly homogeneous in the elementary quantities. This may occur even if the deflator is the Törnqvist index.

---

[6] This conclusion is immediate if one considers that the economic index numbers that are derived from a non-homothetic function could never satisfy, by construction, all the homogeneity requirements.

In searching a way out from this impasse, Diewert (1983a) constructed a cost-based *direct* price index and a *direct* Törnqvist quantity index. This procedure has been supported by Russell's (1983) comments[7]. Diewert (1983b) followed a similar procedure in the theory of output price and quantity changes. However, although both these price and quantity index numbers turn out to satisfy the linear homogeneity requirement, this outcome is achieved at the cost of failing to satisfy the requirements of the factor-reversal test (stating that the price index multiplied by the quantity index should equal the index of total nominal value). Samuelson and Swamy (1974, p. 576) clearly observed: "If, like Pollak, one employs a quantity definition that satisfies Fisher's (*i**) [linear homogeneity test], then [given the imposed linear homogeneity of the price index] one of the other tests, such as (*v**) [weak factor reversal test], will fail in the nonhomothetic case". They spelled out this result even more clearly in another example (p. 577, fn. 10): "Afriat favors the linear Engel-curve approximation: $e(P;Q)=\theta(P)\phi(Q)+\mu(P)$", where the last additive term is a residual not captured by the *linearly homogeneous* price index multiplied by the *linearly homogeneous* quantity index.

In consideration of non-homothetic changes in parameters of the underlying function, Diewert's (1976, pp. 123-124) has also shown that the Törnqvist quantity index can be "exact" for a Malmquist quantity index which is defined by a translog distance function evaluated at the geometric mean of the utilities in the two compared points of observation.. This result has been later generalized by Caves, Christensen, and Diewert (1982, pp. 1409-1413) in their *Translog Identity*. They showed that the Törnqvist index number is "exact" for the geometric mean of two translog functions referred to the two compared points of observation and differing in parameters of their zero- and first-order terms.

The Törnqvist index is, therefore, still regarded as being equally valid for measuring aggregate relative changes in input quantities or prices under alternative assumptions of homothetic *and* non-homothetic changes. Caves *et al*. (1982, p. 1411) claimed: "This result implies that the Törnqvist index is superlative in a considerably more general sense than shown by Diewert. We are not aware of other indexes that can be shown to be superlative in this more general sense". However, since all superlative indexes are supposed to approximate one another numerically, they conclude: "any superlative index (in the sense of Diewert, 1976) will be approximately equal to the geometric mean of two Malmquist indexes based on the translog form". More recently, it has been shown that, in the non-homothetic case, the Törnqvist index number can be "exact" for a more general weighted geometric mean of two translog functions differing in *all* parameters. Contrary to what has been previously contended, a similar result is also valid for all the other indexes that are

---

[7] Russell (1983, p. 237) concludes that the Malmquist quantity index, which Diewert favours because of its intrinsic properties, is in fact the *only* natural counterpart to the widely accepted Konüs cost-of-living index.

encompassed by the quadratic mean-of-order-*r* index number formula (see Milana, 2005).

Diewert (1976, pp. 123-124) and Caves *et. al*. (1982) have explicitely recognized that, in the non-homothetic case, the Translog Identity is not "an if and only if" result, in sense that an aggregator translog function implies that the corresponding "exact" index number formula is Törnqvist, but this could be "exact" for functional forms of the aggregator function other than the translog. The same applies to the other superlative index number formulas which, as shown in Milana (2005), might as well be "exact" also to first-order polynomial functional forms subject to non-homothetic changes. This further weakens the superlativeness character of these index numbers in such cases.

The problems outlined here may worsen in the context of a multilateral comparison when (as it generally happens) the circularity test is not passed with the single bilateral comparisons. Aggregation over inconsistent bilateral comparisons may lead to a systematic bias. This is, in particular, the case of the EKS and CCD methods cited above (see, for example, Neary, 2004, pp. 1414-1416).

Another reason why the chosen index number formula does not satisfy the circularity test is often attributed to the fact it can be exact for some form of approximation to the "true" index. Diewert (1976, pp. 115-117) has also stressed the importance of using the index numbers that are "exact" for quadratic functional forms, which he called *flexible*, since these may be alternatively interpreted as second-order approximations to an arbitrary (twice-differentiable) "true" aggregator function. As shown in Milana (2005), the data on the compared points of observation are not generally consistent with the approximating aggregator function but account for the actual economic behavior. This also prevents the so-called "superlative" index number formula to be exact for a second-order approximating aggregator function. What is really obtained with this formula is an hybrid index number that can be interpreted as a combination of two first-order approximations at the two compared observed points. As Samuelson and Swamy (1974, p. 582) remind us: "[a]pproximations often violate transitivity. For example, 1.01 and .99 are each within 1 percent approximations to 1.0, but *that* does not make them have this property with respect to each other!".

Allen and Diewert (1981, p. 430) had recognized that "in many applications involving the use of cross section data or decennial census data, there can be a tremendous amount of variation in prices or in quantities between the two periods so that alternative superlative index numbers can generate quite different results". In fact, more recently, Hill (2006) has found a large spread in numerical values of alternative superlative index numbers, with the largest and the smallest ones differing by more than 100 per cent using a standard US national data set and by about 300 per cent in a cross-section comparison of countries using an OECD data set. However, it has been surprising to find empirically that the spread between the largest and the smallest Diewert's superlative index numbers may exceed that between the Laspeyres and Paasche indexes. This performance is clearly in contrast with that considered originally by Irving Fisher in identifying his own superlative index numbers.

The Fisher "ideal" index itself (the particular case of the quadratic mean-of-order-*r* index number where $r = 2$, corresponding to the geometric mean of the Laspeyres and Paasche indexes) can be a poor approximation to the "true" index in the non-homothetic case. On this point Samuelson and Swamy (1974, p. 585) clearly wrote: "It is evident that the Ideal index cannot give high-powered approximation to the true index in the general, nonhomothetic case. A simple example will illustrate the degree of this failure […]. [E]ven if (P^1,P^0,P^α) and (Q^1,Q^0) are 'sufficiently close together, it is not true that the Laspeyres [λ_q] and Paasche [π_q] indexes provide two-sided bounds for the true index. In this example, the true index lies outside the [λ_q, π_q] interval!".

We can attempt to decomposed the relative change in the expenditure function as follows

$$\frac{E(p_1,u_1)}{E(p_0,u_0)} = I_p \cdot I_q \tag{10}$$

where I_p and I_q have the meaning of the price and quantity indexes respectively. Following Milana (2005), any price index number that is exact for a continuous aggregator function can be translated into the following form

$$I_p \equiv \frac{\theta + (1-\theta)\sum s_{0i}\dfrac{p_{1i}}{p_{0i}}}{(1-\theta) + \theta\sum s_{1i}\dfrac{p_{0i}}{p_{1i}}} = \left[\sum_{i=1}^{N}\frac{p_{1i}q_{0i}}{p_{0i}q_{0i}}\right]^{\lambda(\theta)} \cdot \left[\sum_{i=1}^{N}\frac{p_{1i}q_{1i}}{p_{0i}q_{1i}}\right]^{1-\lambda(\theta)} \tag{11}$$

where, for t = 0,1, $s_{ti} \equiv p_{ti}\dfrac{\partial E(p_t,u_t)}{\partial p_{ti}} / \sum_j p_{tj}\dfrac{\partial E(p_t,u_t)}{\partial p_{tj}}$

$= p_{ti}q_{ti} / \sum_j p_{tj}q_{tj}$ using Shephard's lemma

and θ is an appropriate parameter whose numerical value depends on the remainder terms of the two first-order approximations of $E(p,u)$ around the base and current points of observations.

The index I_p is linearly homogeneous in p (that is, if $p_1 = \lambda\, p_0$, then $I_p = \lambda$). With $\theta = 0$, it reduces to a Laspeyres index number, whereas, with $\theta = 1$, it reduces to a Paasche index number.

The “true” exact index number, if any, is numerically equivalent to I_p. If the functional form of $E(p_t, u_t)$ is square root quadratic in p, then I_p can be transformed into a Fisher “ideal” index number, which is given by the geometric mean of the Laspeyres and Paasche index numbers. In this case, the index I_p is numerically equivalent to a quadratic mean-of-order-2 index number.

Here, again, the index I_p is invariant with respect to the reference utility level if and only if $E(p,u)$ is homothetically separable and can be written $E(p,u) = e(p) \cdot u$, in which case $s_{ti} = p_{ti} q_{ti} / \sum_j p_{tj} q_{tj} \; = p_{ti} \frac{\partial e(p_t)}{\partial p_{ti}} / \sum_j p_{tj} \frac{\partial e(p_t}{\partial p_{tj}}.$

Moreover, $I_q = [E(p_1, u_1)/E(p_0, u_0)]/I_p$ is the quantity index measured implicitly by deflating the index of the functional value with the price index I_p. It has the meaning of a pure quantity index if and only if I_p is a pure price index.

The parameter θ, however, remains unknown and we cannot rely on the second-order differential approximation paradigm. For this reason, in a previous paper, we concluded that “it would be more appropriate to construct a range of alternative index numbers (including those that are not superlative), which are all equally valid candidates to represent the true index number, rather than follow the traditional search for only *one* optimal formula” (Milana, 2005, p. 44). This conclusion was anticipated long before by Afriat (1977, pp. 107-112):

“The price index P_{10} is determined by the utility *R* and the prices p_0 and p_1. The condition on *R* which permits this determination, for arbitrary p_0 and p_1, is equivalent to the requirement that the utility-cost function admits the factorization into a product

$$E[p,u] = e(p) \cdot f(q) \tag{12}$$

of the price and quantity functions. […] Then the price index, with 0 and 1 as base and current priods, is expressed as ratio

$$P_{10} = P_1 / P_0 \tag{13}$$

of price levels

The conclusion […] is that the price index is bounded by the Paasche and Laspeyres indices. […] The Paasche index does not exceed the Laspeyers index. […] The set of values [of the “true index”] is in any case identical with the Paasche-Laspeyres interval. The “true” points are just the points in that interval and no others; and none is more true than another. There is no sense to a point in the interval being a better approximation to “the true index” than others. There is no proper distinction of ‘constant utility’ indices, since all these points have that distinction”.

The same conclusion is replicated in Afriat (2005, p. *xxiii*):

> “Let us call the *LP* interval the closed interval with *L* [Laspeyers index] and *P* (Paasche index] as upper and lower limits, so the *LP*-inequality is the condition for this to be non-empty. While every true index is recognized to belong to this interval, it can still be asked what points in this interval are true? The answer is all of them, all equally true, no one more true than another. When I submitted this theorem to someone notorious in this subject area it was received with complete disbelief.
>
> “Here is a formula to add to Fisher’s collection, a bit different from the others.
>
> “Index Formula: Any point in the *LP*-interval, if any.”

3. Revealed-preference based tests, recoverability of utility or technology and index number bounds

The difficulty of finding a satisfactory point estimate of the “true” economic price index number based on the adoption of specific formulas has led many authors to find alternative solutions eschewing parametric forms. These alternative solutions are based on the revealed preference theory originally offered by Samuelson (1938a) (1938b)(1938c)(1947)(1948) and Houthakker (1950), first implemented empirically by Houthakker (1963) and Koo (1963).

It was not until the appearance of the fundamental *Afriat’s* (1967b) *theorem* that the revealed preference approach could not be fully exploited empirically starting from the available data. Rather than assuming the existence of a well-behaved continuous differentiable (single-valued) utility function governing the demand generating the observed data, Afriat’s approach is truly constructive and asks whether it is possible to construct a homogeneous utility function starting from the observed data assuming that these are generated by a cost-minimizing demand. Only upper and lower bounds of the numerical values of such utility function can be found, which are themselves possible candidates of such a function. These bounds are piecewise linear (multi-valued) and homogeneous. Hanoch and Rothschild (1972), Diewert (1973), Diewert and Parkan (1983)(1985), and Varian (1982)(1983)(1984)(1985) have tried to clarify many aspects of Afriat’s approach with proposals of practical applications based on linear programming techniques (see also Deaton, 1986, pp. 1796-1798, Russell *et al*., 1998, and the recent survey offered by Varian, 2006).

Afriat’s approach is based essentially on searching for tight bounds of the unknown “true” measure using multiple observations after testing for rationality of the economic behavior generating the available data. This approach is also seen as a necessary preliminary test of the maintained hypotheses of any parametric or non-parametric methods based on the adoption of specific functional forms. In his empirical application of this method to annual U.S. aggregate data on nine consumption categories from 1947 to 1978, Varian (1982) had found that his tests easily satisfied the revealed preference conditions implying that the Engel curves are linear and tastes are homothetic, thus contradicting the expectations of the economic theory. Varian has raised the possibility that these tests are plagued by low power because the effects of changes in total expenditure tend to dominate those in relative prices. In the extreme case where the budget hyperplanes do not intersect at all, the tests have zero power. Many discussions have followed in the literature trying to explain the apparently low power of these tests, which has been found also in subsequent empirical studies, including Manser and McDonald (1988) and Famulari, 1995 (see also the discussions contained in Blundell, *et al.*, 2003 and Blundell, 2005 and the references contained therein).

After having presented an extensive discussion of the economic theory of index numbers at the Department of Applied Economics at the University of Cambridge during the 1950s, Afriat (1960) presented a system of simultaneous linear inequalities in a research memorandum at Princeton University to investigate preference orders which are explanations of expenditure data associating vectors of quantities with prices. He called this system "consistent" if it had solutions. He then established a set of theorems on chaining these inequalities and defined minimized chains of those inequalities which share those solutions. This was just the beginning of an unforeseen development that has brought us to the widespread stream of non-parametric empirical analyses of production and consumption behaviour that is still flourishing in the present days.

In a number of subsequent and remarkable contributions, Afriat (1961)(1962)(1963a)(1963b)(1964) reworked the original axioms defined by Samuelson (1948) and Houthakker (1950) in the field of revealed preference theory, now respectively known under the names of *Weak Axiom of Revealed Preference* (WARP) and *Strong Axiom* (SARP), giving account of how one could recover a set of indifference curves from a finite set of observed data. By using his previous development of minimized chained inequalities, he has been able to strengthen Houthakker's condition and to formulate his *Cyclical Consistency* (CC), later called *Generalized Axiom of Revealed Preference* (GARP) by Varian (1982).

For the purpose of the discussion below, let us consider briefly the following definitions.

Definition 1 (Samuelson's Weak Axiom of Revealed Preference (WARP)). *With a demand function, given some vectors of prices p and chosen bundles q, it is not the case that* $p_0 q_1 \leq p_0 q_0$ *and, simultaneously,* $p_1 q_0 \leq p_1 q_1$ *unless both be equalities.*

Samuelson's original condition, equivalently stated as $p_0q_1 \le p_0q_0$, $p_1\, q_0 \le\ p_1$ $\Rightarrow q_0 = q_1$, has been restated by Afriat (2005, p. 9) with a minor relaxation for a general demand correspondence of a pair of demand elements as follows

$$\text{(condition S)} \qquad p_0q_1 \le p_0q_0, \quad p_1q_0 \le p_1q_1 \;\Rightarrow p_0q_1 = p_0q_0, \quad p_1q_0 = p_1q_1$$

Definition 2 (Directly and transitively revealed preference) We say that q_t is *directly revealed preferred* to a bundle *q* (written q_tR_Dq) if $p_t\,q_t \ge\ p_t\,q$. We say that q_t is *transitively revealed preferred* to a bundle *q* (written q_tRq) if there is some sequence *i,j,k,...,l,m* such that $p_iq_i \ge p_iq_j$, $p_iq_i \ge p_iq_j, \ldots,$ $p_lq_l \ge p_lq_m$. In this case, we say the relation *R* is the *transitive closure* of the relation R_D.

Samuelson's revealed preference was given in the case of two goods only and was primarily graphical. An extension to the general case of multiple goods was successively given by Houthakker (1950), whose conditions can be stated as:

Definition 3 (Houthakker's Strong Axiom of Revealed Preference (SARP)). *If* q_rRq_s *ten it is not the case that* q_sRq_t.

Houthakker's original condition, equivalently stated as $p_iq_j \le p_iq_i$, $p_j\,q_k \le p_k\,q_k$, ..., $p_lq_m \le p_lq_m \Rightarrow q_i = q_j = ... = q_m$, has been restated by Afriat (2005, p. 10) with a minor relaxation for a general demand correspondence as follows

$$\begin{aligned}\text{(condition H)} \qquad & p_iq_j \le p_iq_i, \quad p_jq_k \le p_kq_k, \ldots, \; p_mq_i \le p_mq_m \\ & \Rightarrow p_iq_j = p_iq_i, \quad p_jq_k = p_kq_k, \ldots, \; p_mq_i = p_mq_m.\end{aligned}$$

While the Samuelson-Houthakker approach was intended to establish the recoverability of an *integrable* utility function generating the data, Afriat's approach was directed to ask if a possible non-satiated well-behaved utility function can be constructed that could fit the finite set of observed data. Varian (2006) introduces Afriat's approach in these terms:

"He started with a *finite* set of observed prices and choices and asked how to actually construct a utility function that would be consistent with these choices.[8]

"The standard approach showed, in principle, how to construct preferences consistent with choices, but the actual preferences were described as limits or as solution to some set of partial differential equations.

---

[8] In a footnote, Varian (2006) adds: "I once asked Samuelson whether he thought of revealed preference theory in terms of a finite or infinite set of choices. His answer, as I recall, was: 'I thought of having a finite set of observations … but I always could get more if I needed them!".

“Afriat’s approach, by contrast, was truly constructive, offering an explicit algorithm to calculate a utility function consistent with a finite amount of data, whereas the other arguments were just existence proofs. This makes Afriat’s approach much more suitable as a basis for empirical analysis.

“Afriat’s approach was so novel that most researchers at the time did not recognize its value.”

Another aspect that did not help in making Afriat’s approach immediately understood is the strength itself of his mathematical treatment: an elegant, compact and essential exposition that was not easily readable even by professional experts.

Diewert’s (1973) “clearer” exposition of Afriat’s main results was very useful in bringing the new view to the attention of the profession. In that period a number of innovations in the field of quantitative analysis of production and consumption behaviour were taking place. The so-called flexible forms that could be interpreted as second-order approximation to the true unknown functions were replacing the traditional “rigid” functional forms of integrable functions. Those functional forms seemed to be flexible enough to test consistency of the observed behaviour with theoretical optimizing models and homotheticity and separability hypotheses. It was thought, at the time, that, since the flexible functional forms provide a good (second-order) approximation to the true unknown function, it did not matter much which one was chosen. Surprisingly, it was found that these functional forms loose flexibility if separability is imposed on its parameters, since they become of first-order approximating aggregators of flexible functions of separable variables or remain flexible in first-order approximating aggregators of separable variable. It soon became apparent that the testing procedures were flawed by the fact that it was impossible to disjoint the test of the null hypothesis from that regarding the specific functional forms that were being used (see, for example, Diewert, 1993, p. 15 for references to the literature that discussed this issue).

One promising solution could be to abandon any explicit functional form and to start from the data to see, by means of some non-parametric techniques, if they are compatible with a well-behaved optimizing function (and, more specifically, with a separable and homothetic well-behaved function). As Diewert (1993, p. 28, fn. 24) himself has recalled, Hanoch and Rothschild (1972) have been quick to see that the line of thought and the procedure needed was just of the kind of Afriat’s revealed preference techniques and tried to apply them to the production context. Several years later, Diewert and Parkan (1983)(1985) have tried to develop a similar application by developing an algorithm based on linear programming techniques. In a backward looking perspective, Varian (2006) has noted that these tests would have been performed in any case, also for deciding whether the data could be consistent with the model used before proceeding to the econometric estimation of its parameters or to the construction of index numbers.

Varian (1982) himself has started publishing his own research in this direction. In a later account of the developments that followed, Varian (2006) give us the following information:

"In 1977, during a visit to Berkeley, Andreu Mas-Collel pointed me to Diewert's (1973) exposition of Afriat's analysis, which seemed to me a more promising basis for empirical applications.

"Diewert (1973) in turn led to Afriat (1967). I corresponded with Afriat during this period, and he was kind enough to send me a package of his writing on the subject. His monograph Afriat (1987) offered the clearest exposition of his work in this area, though, as I discovered, it was not quite explicit enough to be programmed into a computer".

"I worked on reformulating Afriat's argument in a way that would be directly amenable to computer analysis. While doing this, I recognized that Afriat's conditions of "cyclical consistency" was *basically* equivalent to Strong Axiom. Of course, in retrospect this had to be true since both cyclical consistency and SARP were necessary and sufficient conditions for utility maximization" (italics added).

By recognizing that the most convenient result for empirical work comes from Afriat's approach, Varian (1982) restated his "cyclical consistency" for a set of observations of prices p_t and quantities q_t, for *t* = 1,…,*T*, with the following:

Definition 4 (Generalized Axiom, or Test, of Revealed Preference (Varian, 1982, 1983)) *The data* (p_t, q_t) *satisfy the Generalized Axiom of Revealed Preference (GARP) if* $q_t R q_s$ *implies* $p_s q_s \le p_s q_t$.

The difference between Afriat's CC (or Varian's GARP) and Houthakker's SARP is that the *strong* inequality in SARP (represented using the notation <) becomes a *weak* inequality in GARP (represented using the notation $\le$), thus allowing for multivalued demand functions and flat indifference curves in some range of values, which turns out to be crucial for a direct application in empirical work.

The main result, as restated by Varian (2006), is the following

Theorem 1.1 (Varian's (1983, p. 100)(2007) formulation of Afriat's Theorem). Given some choice data (p_t, q_t) *for t = 1,…,T, the following conditions are equivalent.*

1. *There esists a nonsatiated utility function u(x) that rationalizes the data in the sense that for all t,* $u(x_t) \ge u(x)$ *for all x such that* $p_t x_t \ge p_t x$.
2. *The data satisfy GARP.*
3. *There is a positive solution* (u_t, λ_t) *to the set of linear inequalities*

$$u_t \le u_s + \lambda_s p_s (x_t - x_s) \qquad \textit{for all } s,t.$$

4. *There exists a nonsatiated, continuous, monotone, and concave utility function $u(x)$ that rationalizes the data.*

Fostel, Scarf and Todd (2004) give a somewhat more transparent restatement of Afriat's Theorem as follows:

Theorem 1.2 (Fostel, Scarf and Todd's (2004) formulation of Afriat's Theorem). *If the data set D* $[\equiv(p_t, x_t)]$ *satisfies GARP then there exists a piecewise linear, continuous, strictly monotone and concave utility functions that generates the observations.*

This piecewise linear (multivalued) utility function should be contrasted with the continuously differentable (singlevalued) utility function that Samuelson and Houthakker had previously in mind. Quoting Afriat (1964),

"[T]he data could be assumed infinite but not necessarily complete. Also even with completeness, the usual assumption of a single valued demand system could be omitted. Or, if a single valued function is assumed, the Lipschitz-type condition assumed by Uzawa (1960) and, therefore, also the differentiability assumed by other writers can be dropped".

The original statement of this theorem was the following:

Theorem 1.3 (Afriat's Theorem in the original statement of Afriat (1964)). *The three conditions of cyclical, multiplier and level consistency on the cross-structure of an expenditure configuration are all equivalent, and are implied by the condition of utility consistency for the configuration.*

Afriat's cyclical consistency (CC) corresponds, in this context, to Varian's GARP, whereas *multiplier consistency* (MC) is equivalent to condition no. 3 in Varian's restatement of the theorem and *level consistency* (LC) is the following condition:

$$\lambda_r p_r(x_s - x_r) + \lambda_s p_s(x_t - x_s) + \ldots + \lambda_q p_q(x_q - x_r) \geq u_s - u_r + u_t - u_s + \ldots + u_q - u_r = 0$$

which means that no "technical inefficiency" affects the data.

The construction of the bounds of economic price and quantity indexes requires that the underlying preferences be homothetic. Under this condition, Afriat's theorem, reformulated in Afriat (1981, p. 145), has led Varian (1983)(2007) to provide the following

Definition 5 (Homothetic Axiom of Revealed Preference (HARP)): *A set of data* (p_t, q_t) *for* $t = 1,\ldots,T$ *satisfy the Homothetic Axiom of Revealed Preference (HARP) if, for every sequence i,k,…,l,j,* (condition K) $$\frac{p_i q_k}{p_i q_i}\frac{p_k q_l}{p_k q_k}...\frac{p_l q_j}{p_l q_l} \geq \frac{p_j q_j}{p_j q_i}$$

The left- and right-hand sides of the foregoing inequality are, respectively, a *chained* Laspeyres index and a *direct* Paasche index of quantities.

We may note that condition K is equivalent to the following condition that is expressed in terms differences rather than ratios:

(condition K') $$p_i(\tilde{q}_k - \tilde{q}_i) + p_k(\tilde{q}_l - \tilde{q}_k) + ... + p_m(\tilde{q}_j - \tilde{q}_m) \geq p_j(\tilde{q}_j - \tilde{q}_i)$$

where the hat $\tilde{}$ means that the vector of variables is normalized (by dividing each element, for example, by the first one). The left- and right-hand sides of the foregoing inequality are, respectively, a *chained* Laspeyres-type indicator and a *direct* Paasche-type indicator of quantities.

The condition K and K' are obviously a strengthening of condition H,

$$\text{Condition K and K'} \Rightarrow \text{Condition H}$$

The corresponding inequality of price indexes obtained as the ratio between the index of the value of total expenditure and the indexes of foregoing aggregate quantities is therefore

$$\frac{p_k q_i}{p_i q_i}\frac{p_l q_k}{p_k q_k}...\frac{p_j q_l}{p_l q_l} \leq \frac{p_j q_j}{p_i q_j} \tag{14}$$

where the left-hand side of the foregoing inequality is a chained Paasche index and the right-hand side is a direct Laspeyres index of the aggregate of prices p_m relative to the aggregate of quantities p_i. And in terms of differences

$$(p_k - p_i)\tilde{q}_i + (p_l - p_k)\tilde{q}_k + ... + (p_j - p_m)\tilde{q}_m \leq (p_j - p_i)\tilde{q}_j \tag{15}$$

Using a more synthetic notation, we have, respectively,

$$K_{ki}K_{lk}...K_{jl} \leq L_{ji} \qquad \text{(in terms of ratios)}$$

and

$$\Gamma_{ki} + \Gamma_{lk}... + \Gamma_{jl} \leq \Lambda_{ji} \qquad \text{(in terms of differences)}$$

where

$L_{rs} \equiv \dfrac{p_r q_s}{p_s q_s}$ and $\Lambda_{rs} \equiv (p_r - p_s)\tilde{q}_s$ denote, respectively, a Laspeyres price index and a Laspeyres-type price indicator, and

$K_{rs} \equiv \dfrac{p_r q_r}{p_s q_r}$ and $\Gamma_{rs} \equiv (p_r - p_s)\tilde{q}_r$ denote, respectively, a Paasche price index and a Paasche-type price indicator

with $L_{rs} = \dfrac{1}{K_{sr}}$ and $\Lambda_{rs} = -\Gamma_{sr}$

Dividing both sides of the price-index inequality by $(K_{ki}K_{lk}...K_{jl}L_{ji})$ yields

$$\text{(condition L)} \qquad L_{ik}L_{kl}...L_{lj} \geq K_{ij} \qquad (16)$$

and subtracting $(\Gamma_{ki} + \Gamma_{lk}... + \Gamma_{jl} + \Lambda_{ji})$ from both sides of the price-indicator inequality yields

$$\text{(condition L')} \qquad \Lambda_{ki} + \Lambda_{lk}... + \Lambda_{jl} \geq \Gamma_{ji} \qquad (17)$$

Condition L, L’, K, and K’ are equivalent:

$$\text{Condition K and K'} \Leftrightarrow \text{Condition L and L'}$$

We note that each of these conditions K, L, K’, and L’ is necessary and sufficient for HARP. This definition is consistent with the necessary and sufficient condition given by Afriat (1977) for homogeneity of the utility function for the sequence *i,j* implying also the Hicks (1956, p. 181)-Afriat (1977) Laspeyres-Paasche inequality (LP-inequality) condition for a homogeneous utility function

$$\text{(LP-inequality condition)} \qquad L_{ij} \geq K_{ij} \qquad (18)$$

which, we may note, is equivalent to

$$\Lambda_{ij} \geq \Gamma_{ij} \qquad (19)$$

Afriat (2005, p. 9) noted that “The [LP] inequality is a strenghtening of revealed preference consistency test which assures the data to fit some utility. For fit with not any utility but a utility restricted to be homogeneous, or conical, as required for dealing with a price index, a stricter test would be needed—in fact, the LP-inequality!

"The LP-inequality has gained significance, where it is identified as the homogeneous counterpart of Samuelson's revealed preference condition, for the data to fit a homogeneous utility, as required for dealing with a price index".

The S condition regards the test of the data for consistency with a rational behavior governed by a utility that is not necessarily homothetic (homogeneous), whereas the LP-inequality restrict the test to the special case of a homogeneous utility, that is

$$\text{Not condition S} \quad \rightarrow \quad \text{Not LP-inequality condition}$$

and, equivalently,

$$\text{LP inequality} \rightarrow \quad \text{condition S}$$

In the case of only two demand observations, LP-inequality condition coincides with condition K for HARP. As Afriat (2005, p. 10, fn. 6) put it, "[h]ere there is appeal to a 'homogeneous' counterpart of so-called Afriat's Theorem, with a test that strengthens Houthakker's, or rather, to the special case with two demand observations, where that stronger test [i.e. SARP test] reduces to the LP-inequality."

We also note that

$$K_{ij} \leq \frac{P_i}{P_j} \leq L_{ij} \tag{20}$$

where P_i and P_j are price level solutions for the demand observations *i* and *j*. These price level solutions exist if and only if the LP-inequality condition is satisfied.

Analogously,

$$\tilde{\Gamma}_{ij} \leq (P_i - P_j) \leq \tilde{\Lambda}_{ij} \tag{21}$$

where $\tilde{\Gamma}_{ij} \equiv \Gamma_{ij} / \tilde{Q}_i$ and $\tilde{\Lambda}_{ij} \equiv \Lambda_{ij} / \tilde{Q}_j$ with $\tilde{Q}_t \equiv p_t \tilde{q}_t / P_t$.

The above conditions can be reformulated in terms of differences rather than ratios.

In the case of several demand observations, the LP inequality could be extended in an appropriate way in order to take account of all those observations simultaneously. Following Afriat (1981, p. (1984, p. 47)(2005, p. 167), let us define, for all the demand observation pairs *i*,*j*,

$$M_{ij} \equiv \min_{kl\ldots m} L_{ik} L_{kl} \ldots L_{mj} \qquad \text{(minimum chained Laspeyres price index number)}$$

$$\Upsilon_{ij} \equiv \min_{kl\ldots m} \Lambda_{ik} + \Lambda_{kl} \ldots + \Lambda_{mj} \text{ (mimimum chained Laspeyres-type price indicator)}$$

and

$$H_{ij} \equiv \max_{kl\ldots m} K_{ik} K_{kl} \ldots K_{mj} \qquad \text{(maximum chained Paasche price index number)}$$
$$= \frac{1}{M_{ji}}$$
$$\Psi_{ij} \equiv \max_{kl\ldots m} \Gamma_{ik} + \Gamma_{kl} \ldots + \Gamma_{mj} \text{ (maximum chained Paasche-type price indicator)}$$

where the conditions for existence of the solution are, respectively, that

$$L_{ik} L_{kl} \ldots L_{mj} \geq K_{ij} \quad \text{and} \quad \Lambda_{ik} + \Lambda_{kl} \ldots + \Lambda_{mj} \geq \Gamma_{ij} \tag{22}$$

and

$$K_{ik} K_{kl} \ldots K_{mj} \leq L_{ij} \quad \text{and} \quad \Gamma_{ik} + \Gamma_{kl} \ldots + \Gamma_{mj} \leq \Lambda_{ij} \tag{23}$$

Therefore, in the case of multiple demand observations, the LP-inequality for recoverability of a homogeneous utility function becomes the following "optimized" Chained Laspeyres-Paasche (CLP) inequality

(CLP-inequality condition) $M_{ij} \geq H_{ij}$ (tight bounds of the "true" price index number)

and, equivalently, $\Upsilon_{ij} \geq \Psi_{ij}$ (tight bounds of the "true" price indicator).

As shown by Afriat (1981, p. 154-155)(1984, p. 48), the derived chained Laspeyres and Paasche indexes satisfy the *triangle inequalities*

$$M_{it} M_{tj} \geq M_{ij} \quad \text{and} \quad \Upsilon_{it} + \Upsilon_{tj} \geq \Upsilon_{ij} \quad \text{for every } i,j,t \tag{24}$$

$$H_{it} H_{tj} \leq H_{ij} \quad \text{and} \quad \Psi_{it} + \Psi_{tj} \leq \Psi_{ij} \quad \text{for every } i,j,t \tag{25}$$

given that $\frac{1}{M_{ti}} \frac{1}{M_{jt}} \leq \frac{1}{M_{ji}}$ and $-\Upsilon_{ti} - \Upsilon_{jt} \leq -\Upsilon_{ji}$ for every i, j, t.

We can now state the following finite test:

Definition 6 (CLP-inequality): *A set of data* (p_t, q_t) *for* $t = 1,...,T$ *is said to satisfy the chain LP-inequality (CLP-inequality) if, for every pair of demand observations (i,j) and every possible sequence i,k,l,…,m,j*

$$\text{(CLP-inequality)} \quad \underset{k,l,...,m}{\text{Min}} \frac{p_i q_k}{p_i q_i} \frac{p_k q_l}{p_k q_k} ... \frac{p_m q_j}{p_m q_m} \ge \underset{k,l,...,m}{\text{Max}} \frac{p_k q_k}{p_k q_i} \frac{p_l q_l}{p_l q_k} ... \frac{p_j q_j}{p_j q_m}$$

We may note that this condition implies

$$\underset{k,l,...,m}{\text{Min}}\, p_i(\tilde{q}_k - \tilde{q}_i) + p_k(\tilde{q}_l - \tilde{q}_k) + ... + p_m(\tilde{q}_j - \tilde{q}_m) \ge \underset{k,l,...,m}{\text{Max}}\, p_k(\tilde{q}_k - \tilde{q}_i) + p_l(\tilde{q}_l - \tilde{q}_k) + ... + p_j(\tilde{q}_j - \tilde{q}_m) \tag{43}$$

We also note that the CLP-inequality condition is more stringent than conditions K, LP-inequality:

$$\overbrace{\underbrace{K_{ij} \le \overbrace{H_{ij} \le M_{ij}}^{\text{CLP-inequality}} \le (L_{ik} L_{kl} ... L_{mj}, \forall k,l,...m)}_{\text{Condition K for HARP}} \le L_{ij}}^{\text{LP-inequality}} \tag{44}$$

The CLP-inequality implies the tightest upper and lower bounds for the “true” price ratio P_i / P_j, that is

$$H_{ij} \le \frac{P_i}{P_j} \le M_{ij} \tag{45}$$

with P_i and P_j being the price level solutions for the demand observations *i* and *j* in a context of multiple demand observations.

Also CLP-inequality implies the tightest upper and lower bounds for the price and quantity indicators:

$$\tilde{\Psi}_{ij} \le (P_i - P_j) \le \tilde{\Upsilon}_{ij} \tag{46}$$

where $\tilde{\Psi}_{ij} \equiv \Psi_{ij} / \tilde{Q}_i$ and $\tilde{\Upsilon}_{ij} \equiv \Upsilon_{ij} / \tilde{Q}_j$, with $\tilde{Q}_t \equiv p_t \tilde{q}_t / P_t$.

The CLP-inequality condition has been implicitly defined by Afriat (1984, p. 47)(Afriat, 2005, p. 167), who has claimed: “The availability of more data has removed some indeterminacy and narrowed the limits. These numbers [*i.e.* the traditional bilateral Laspeyres index numbers] can fall outside limits now effective, and then they are not true indices themselves, unlike the Laspeyres index for two isolated periods. The new limits are obtained by a generalized formulae or algorithm involving all the data simultaneously, unlike the Laspeyres and all price index formulae of the type recognized by Fisher. The result is not even conventionally algebraical in the way usually required for a price index." [9]

From the CLP-inequality condition

$$M_{ij} \le L_{ij} \quad \text{and} \quad \Gamma_{ij} \le \Lambda_{ij} \tag{47}$$

and, since $L_{ii} = 1$ and $\Lambda_{\text{ii}} = 0$ by construction, then these imply

$$M_{ii} \le 1 \quad \text{and} \quad \Lambda_{\text{ii}} \le 0 \tag{48}$$

whereas, from the condition for existence, $K_{ij} \le L_{ik} L_{kl} ... L_{mj}$ and $\Gamma_{ij} \le \Lambda_{ik} + \Lambda_{kl} ... + \Lambda_{mj}$ for all $k, l, ...m,$ we derive

$L_{ik} L_{kl} ... L_{mj} L_{ji} \ge 1,$ given that $K_{ij} = 1/L_{ji}$, and $\Lambda_{ik} + \Lambda_{kl} ... + \Lambda_{mj} + \Lambda_{ji} \ge 0$ given that $\Gamma_{ij} = -\Lambda_{ji}$, that is

$$M_{ii} \ge 1 \qquad \text{where } M_{ii} \equiv \text{Min}_{k,l,...j} L_{ik} L_{kl} ... L_{mj} L_{ji} \tag{49}$$

and

$$\Lambda_{ii} \ge 0 \qquad \text{where } \Lambda_{ii} \equiv \text{Min}_{k,l,...j} \Lambda_{ik} + \Lambda_{kl} ... + \Lambda_{mj} + \Lambda_{ji} \tag{50}$$

Therefore, the opposite inequalities involving M_{ii} and Λ_{ii} imply, respectively,

$$1 \le M_{ii} \le 1 \qquad \text{and} \qquad 0 \le \Lambda_{ii} \le 0 \tag{51}$$

hence

$$M_{ii} = 1 \qquad \text{and} \qquad \Lambda_{ii} = 0 \tag{52}$$

Any single element $M_{ii} < 1$ and $\Lambda_{ii} < 0$ signals, therefore, the inconsistency of the system. As shown in Afriat and Milana (2008), a correction of the solutions is determined by finding a critical efficiency parameter ε^* such that $0 < \varepsilon^* \le 1$, where $\varepsilon^* = 1$ means full cost-efficiency, so that the system

---

[9] This test has been described and applied empirically by Afriat and Milana (2008).

$$L_{ij} / \varepsilon \geq \frac{P_i}{P_j} \tag{53}$$

is consistent if and only if $\varepsilon \leq \varepsilon^*$. Then, with the adjusted Laspeyres matrix

$$L_{ij}^* \equiv L_{ij} / \varepsilon \qquad \text{and} \qquad \Lambda_{ij}^* \equiv \Lambda_{ij} - \eta(\varepsilon) \quad (i \neq j) \tag{54}$$

the system

$$L_{ij}^* \geq \frac{P_i}{P_j} \qquad \text{and} \qquad \Lambda_{ij}^* \geq P_i - P_j \tag{55}$$

is consistent, and with

$$M^* = (L^*)^n \tag{56}$$

the basic price level solutions and price indices can be obtained from M^*.

The picture of the relations between the tests for revealed preference and existence and bounds of the "true" index numbers can be completed as in Table 1.

We may recall that the conditions S and H for the respective Samuelson's and Houthakker's Revealed Preferences and their more general version defined by Afriat (1967b), now known as GARP after Varian (1982a)(1982b)(1983), are implied but do not imply the LP-and CLP-inequalities, which are necessary and sufficient for a price level to exist.

The broader limits of indeterminacy in the HARP test for the constructability of a homogenous utility function governing the ordering of demand observations may help explaining the apparently paradoxical results obtained in the literature in the application of this test, which was seldom violated in contrast with the expectations of economic theory. We can see now that the HARP test can be still satisfied opening the way to the constructability of a possible homothetic (homogeneous) utility function rationalizing the data but falling outside the tighter bounds of the CLP-condition for the existence of aggregates of all the observed prices and quantities.

Table 1. Relations between tests for consistency with rational behaviour and the existence and bounds of a "true" economic index number

General case: Testing for rational behaviour governed by a well-behaved utility function

WARP (Samuelson, 1948) testing 2 demand observations for consistency with a single-valued utility function).

↑

SARP (Houthakker, 1950) testing $N \geq 2$ demand observations for consistency with a *single-valued* utility function).

↑

GARP (Afriat, 1967, Varian, 1982) testing $N \geq 2$ demand observations for consistency with a *multi-valued* (*piecewise linear*) utility function.

↑ **Homothetic case: Testing for rational behaviour governed by a well behaved *homothetic* (*homogeneous*) utility function**

LP-inequality (Hicks, 1956, Afriat,) testing $N \geq 2$ demand observations for consistency with a homothetic (homogeneous) utility function and the existence of a price (and quantity) index.

If $N > 2$, ↑ or, if $N = 2$, ↕

HARP (Afriat, 1981, 1984, Varian, 1983) testing for $N \geq 2$ demand observations for consistency with a multi-valued (piecewise linear) homothetic (homogeneous) utility function.

↑

CLP-inequality (Afriat, 1984) testing $N > 2$ demand observations for consistency with a homothetic (homogeneous) utility function and the existence of a price (and quantity) index.

Moreover, the system of inequalities

$$M_{ij} = \frac{1}{H_{ji}} \geq P_{ij} \qquad \text{for every } i,j \tag{26}$$

where

$$P_{ij} = \frac{P_i}{P_j} \qquad \text{for every } i,j$$

has solutions in terms of price levels. Analogously, the system of inequalities

$$\tilde{\Lambda}_{ij} = -\tilde{\Gamma}_{ji} \geq \Delta_{ij} \qquad \text{for every } i,j \tag{27}$$

where

$$\Delta_{ij} = P_i - P_j \qquad \text{for every } i,j$$

has the same solutions in price levels. The price level P_t can be interpreted as values of the aggregator price function

$$P_t = e(p_t) \qquad \text{every } i,j \tag{28}$$

where $e(p)$ is the "true" aggregator function of prices.

Accordingly, the system of inequalities

$$H_{ij} = \frac{1}{M_{ji}} \leq \frac{P_i}{P_j} \quad \text{for every } i,j \tag{32}$$

has solutions that may be interpreted in terms of price levels. Analogously, the system of inequalities

$$\tilde{\Gamma}_{ij} = -\tilde{\Lambda}_{ij} \leq (P_i - P_j) \text{ for every } i,j \tag{33}$$

has the same solutions in price levels.

There remains, however, a certain degree of indeterminacy regarding the aggregate price levels. While the "true" measure remains unknown, we have tight upper and lower bounds of its possible values. The remaining question regards the transitivity requirement. In both matrices M and H we have noticed the triangular inequalities $M_{it}M_{tj} \geq M_{ij}$ and $H_{it}H_{tj} \leq H_{ij}$ for every *i,j,* and *t*. By contrast, any true aggregate economic measure defined in terms of ratios or differences and its bounds should be transitive, so that $P_{it}P_{tj} = P_{ij}$ and $\Delta_{it} + \Delta_{tj} = \Delta_{ij}$.

In the general case, while remaining in a range of indeterminacy, we can calculate alternative bounds of price index numbers and indicators using the tight upper and lower level solutions. If, without any loss of generality, the aggregate price levels are assumed to be normalized so that $P_1 = 1,$ then

$$\widehat{P}_i = \max_t \; M_{it} / M_{1t} = \max_t \; M_{it} \cdot H_{t1} \qquad \text{for all } i\text{'s} \qquad (57)$$

$$\widehat{\Delta}_i = \max(\Upsilon_{it} - \Upsilon_{1t}) = \max(\Upsilon_{it} + \Psi_{t1}) \qquad \text{for all } i\text{'s} \qquad (57')$$

where $\widehat{P}_i = \widehat{e}(p)$ with

$$\widehat{e}(p) \equiv \inf_x \left\{ px : p_t x_t \geq e(p); x_t = q_t / f(q_t) \text{ for all } t \right\}$$

being the "over-cost" aggregator function of price levels.

Analogously,

$$\breve{P}_i = \min_t H_{it} / H_{1t} = \min_t H_{it} \cdot M_{t1} \qquad \text{for all } i\text{'s} \qquad (58)$$

$$\breve{\Delta}_i = \min_t (\Psi_{it} - \Psi_{1t}) = \min_t (\Psi_{it} + M_{t1}) \quad \text{for all } i\text{'s} \qquad (58')$$

where $\breve{P}_i = \breve{e}(p)$ with

$$\breve{e}(p) = \sup_x \left\{ px : p_t x_t \leq e(p); \; x_t = q_t / f(q_t) \text{ for all } t \right\} \qquad (36)$$

being the "under-cost" aggregator function of price levels.

The "over-cost" and "under-cost" aggregators of prices $\widehat{e}(p)$ and $\breve{e}(p)$ are concave conical functions which have polyhedral and polytope forms, respectively. They are conjugate dual to "under-cost" and "over-cost" aggregators of quantities $\breve{f}(q) = p \cdot q / \widehat{e}(p)$ and $\widehat{f}(q) = p.q / \breve{e}(p)$ that are concave conical functions with polytope and polyhedral forms, respectively. These functions are such that $\breve{e}(p) \leq \widehat{e}(p)$ and $\breve{f}(q) \leq \widehat{f}(q)$.

The chain-consistent bounds of aggregate price measure are therefore obtained as

$$\widehat{P}_{ij} = \widehat{P}_i / \widehat{P}_j \quad \text{and} \quad \widehat{\Delta}_{ij} \equiv \widehat{\Delta}_i - \widehat{\Delta}_j \quad \text{for every } i,j \qquad (59)$$

$$\breve{P}_{ij} = \breve{P}_i / \breve{P}_j \quad \text{and} \quad \breve{\Delta}_{ij} \equiv \breve{\Delta}_i - \breve{\Delta}_j \quad \text{for every } i,j \qquad (60)$$

The price index numbers defined in terms of ratios satisfy all *Fisher's tests*:

$\overset{\frown}{P}_{ii} = 1$ and $\breve{P}_{ii} = 1$ for every i *Identity test*

$\overset{\frown}{P}_{ij} = \lambda$ and $\breve{P}_{ij} = \lambda$ if $p_i = \lambda p_j$ *General mean of price relatives test* (linear homogeneity in price levels) from which the identity test can be derived as a special case with $\lambda = 1$)

$\overset{\frown}{P}_{ij}\overset{\frown}{P}_{ji} = 1$ and $\breve{P}_{ij}\breve{P}_{ji} = 1$ for every i, j *Time-reversal test*

$\overset{\frown}{P}_{ij}\overset{\frown}{P}_{jk} = \overset{\frown}{P}_{ik}$ and $\breve{P}_{ij}\breve{P}_{jk} = \breve{P}_{ik}$ for every i, j, k *Chain (or Circular-reversal) test*

$\overset{\frown}{P}_{ij} = \overset{\frown}{P}_{ij}^*$ and $\breve{P}_{ij} = \breve{P}_{ij}^*$ where $p_t^* = \alpha p_t$ and $q_t^* = q_t / \alpha$ for $t = i, j$ *Dimensional invariance test*

$\overset{\frown}{P}_{ij}\breve{Q}_{ij} = E_i / E_j$ and $\breve{P}_{ij}\overset{\frown}{Q}_{ij} = E_i / E_j$ for every *i,j* *(Weak) factor-reversal test*[10]

Fisher's tests, originally defined for index numbers in terms of ratios, are equivalent to the following tests, valid for indicators in terms of differences:

Identity test: $\overset{\frown}{\Delta}_{ii} = \breve{\Delta}_{ii} = 0$; *General mean of price relative test*: $\overset{\frown}{\Delta}_{Pij} = \breve{\Delta}_{Pij} = (\lambda - 1)$ if $p_i = \lambda p_j$; *Time reversal test*: $\overset{\frown}{\Delta}_{ij} + \overset{\frown}{\Delta}_{ji} = 0$; $\breve{\Delta}_{Pij} + \breve{\Delta}_{Pji} = 0$; *Chain (or Circular reversal) test*: $\overset{\frown}{\Delta}_{ij} + \overset{\frown}{\Delta}_{jk} = \overset{\frown}{\Delta}_{ik}$; $\breve{\Delta}_{ij} + \breve{\Delta}_{jk} = \breve{\Delta}_{ik}$; *Dimensional invariance test (if also prices are normalized)*:

$\overset{\frown}{\Delta}_{ij} = \overset{\frown}{\Delta}_{ij}^*$ and $\breve{\Delta}_{ij} = \breve{\Delta}_{ij}^*$ where $p_t^* = \alpha p_t$ and $q_t^* = q_t / \alpha$ for $t = i, j$; *(Weak) factor-reversal test*: $\overset{\frown}{\Delta}_{ij} + \breve{\Delta}_{Qij} = \tilde{E}_i - \tilde{E}_j$ and $\breve{\Delta}_{ij} + \overset{\frown}{\Delta}_{Qij} = \tilde{E}_i - \tilde{E}_j$ for every *i,j.*

This result seems to contrast the conclusions derived from Frisch's "impossibility theorem" in index number theory and, more recently from those of other authors (see,

---

[10] Samuelson and Swamy (1974, p. 575) have introduced the concept of the *weak* factor-reversal test, as opposed to the *strong* factor-reversal test: "we drop the *strong* requirement that the *same* formula should apply to *q* s to *p*. A man and wife should be properly matched; but that does not mean I should marry my identical twin!".

for example, Van Veelen, 2002), but is perfectly in line with Samuelson and Swamy (1974, p. 566), who have claimed: "[a]lthough Ragnar Frisch (1930) has proved that, when the number of goods exceeds unity, it is impossible to find well-behaved formulae that satisfy *all* of these Fisher criteria, we derive here canonical index numbers of price and quantity that do meet the spirit of all of Fisher's criteria in the only case in which a single index number of the price of cost of living makes economic sense—namely, the ("homothetic") case of unitary income elasticities in which at all levels of living the calculated price change is the same. This seeming contradiction with Frisch is possible because the price and quantity variables are not here allowed to be arbitrary independent variables, but rather are constrained to satisfy the observable demand functions which optimize well-being" (emphasis in the original text).

In the practical case, we cannot deal with the *single* "true" index number just because this remains unknown, but we can construct two bounds (the tightest upper and lower bounds) of the closed set of possible numerical values of this index number, if the conditions of its existence are satisfied. The toll we pay for satisfying all Fisher's tests and overcome the "impossibility theorem" is to deal with two bound estimates rather then an "ideal" single measure, which ends unavoidably to fail to satisfy at least one of those requirements.

Afriat's method is to find whether a well behaved utility function can be reconstructed that is consistent with the finite set of observed choices satisfying GARP. However, this utility is not unique. There are generally other utility functions and the *recoverability problem* becomes how to reconstruct the entire set of these utility functions that would fit the observed data simultaneously.

For any given q_0, there is the set of q's that are revealed preferred to q_0 ($RP(q_0)$) and set of q's that are revealed worse than q_0 ($RW(q_0)$). A simple example is given in Figure 1. The area corresponding to the set of possible utility functions which satisy GARP is that which does not belong to $RP(q_0)$ and $RW(q_0)$.

The necessary and sufficient condition for the existence of a price and quantity aggregate measure is that the observed data are consistent with homothetic preferences. Following Keynes' (1930, pp. 105-106) "method of limits", as re-exposed by Afriat (1977, pp. 108-115), we may ask whether it is possible to identify the area corresponding to the set of money metric utility functions passing through the reference point. The same observations considered in Figure 1 are shown in Figure 2, where this area is restricted between the upper and lower bounds of the value of the expenditure function passing through the point q_0. These bounds are obtained by inflating the base value of the expenditure function by means of the Laspeyres and Paasche price indexes, respectively. We may note that these bounds are generally tighter than the limits represented with revealed-preference methods (as shown in Fugure 1).

We may also note that Keynes’ method of limits can be applied to the current period perspective, using the current period situation as in Figure 3 rather than that of the base period as in Figure 2. We may note that the relevant parts of the upper and lower bounds from the current period viewpoint are, respectively, the same as those of the lower and upper bounds from the base period viewpoint. The ratios of values corresponding to the Laspeyres and Paasche indexes that are constructed on bounds from one perspective are reciprocal to the Paasche and Laspeyres indexes that are constructed on the bounds from the other perspective.

The inclusion of a third point of observation, as that between the two former points in Figure 4, permits us to track the isoquant or indifference curve using hypothetical budget lines passing through one point along an approximating path followed by the optimal chained Laspeyres or Paasche indexes. This constitutes the upper and lower bounds which are tighter than the direct bilateral Laspeyres and Paasche indexes.

“Revealed-preference” method (as outlined by Varian, 2006)

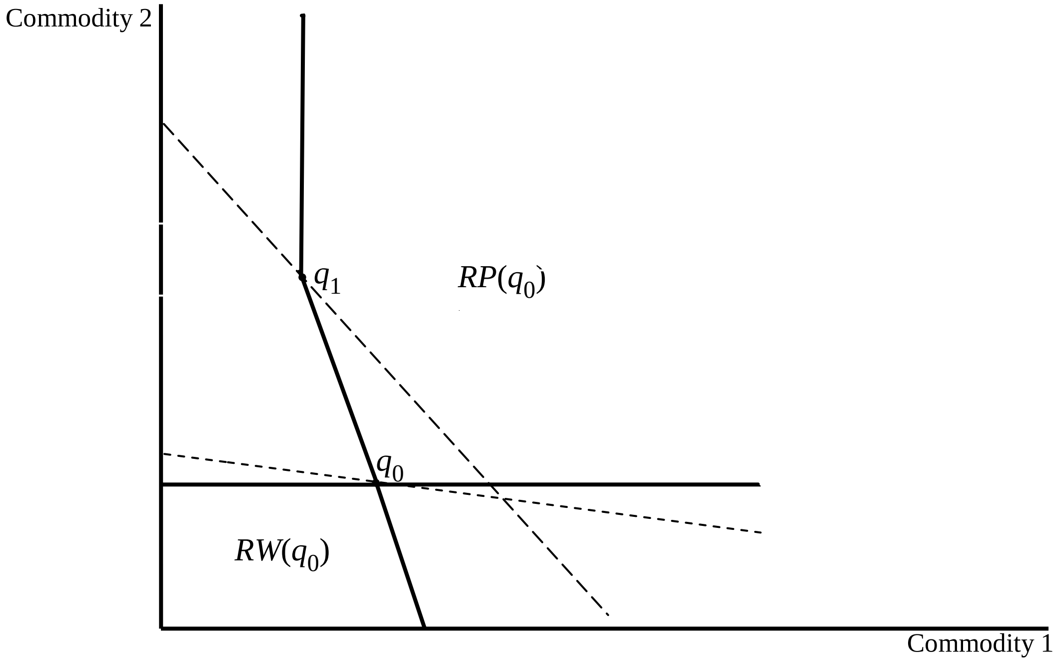


Figure 1: $RP(q_0)$ and $RW(q_0)$: simple case of “revealed preference test.

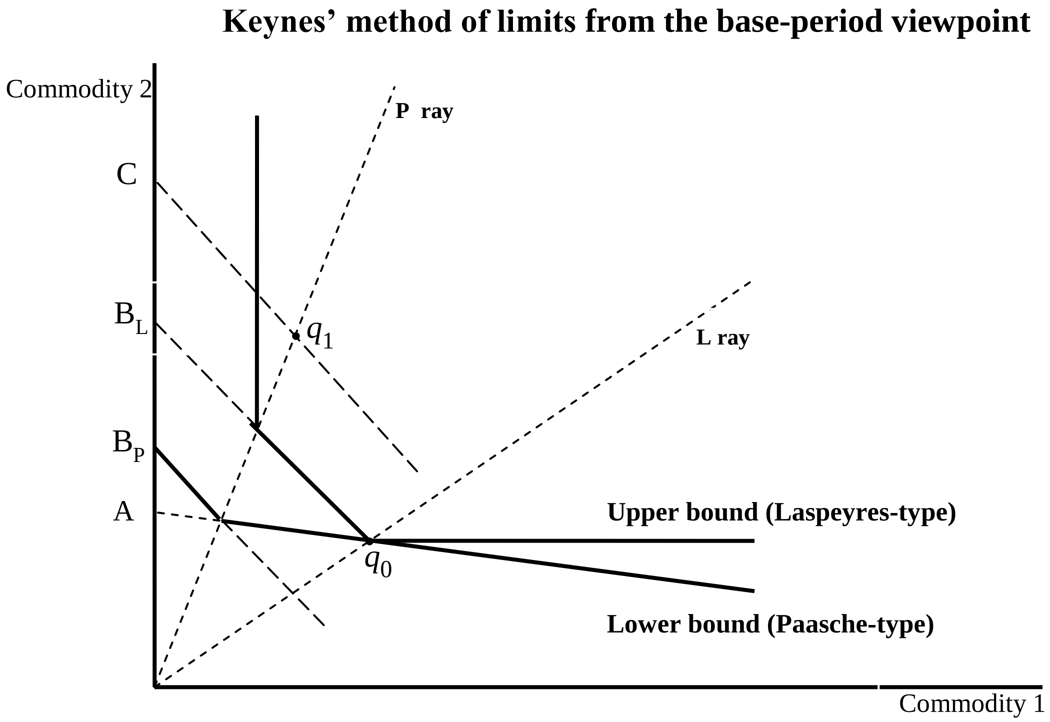

Keynes' method of limits from the base-period viewpoint

Figure 2: Laspeyres- and Paasche-type bounds based on q_0

AC: Observed increase in nominal expenditure in terms of commodity 2 (at the relative prices represented by the slope of Aq_0 and Cq_1, respectively);

AB_P: Price component of the increase in nominal expenditure measured with the direct Paasche index number;

AB_L: Price component of the increase in nominal expenditure measured with the direct Laspeyres index number;

B_PC: Quantity component of the increase in nominal expenditure measured with the implicit Paasche index number;

B_LC: Quantity component of the increase in nominal expenditure measured with the implicit Laspeyres index number;

Keynes' method of limits from the current-period viewpoint

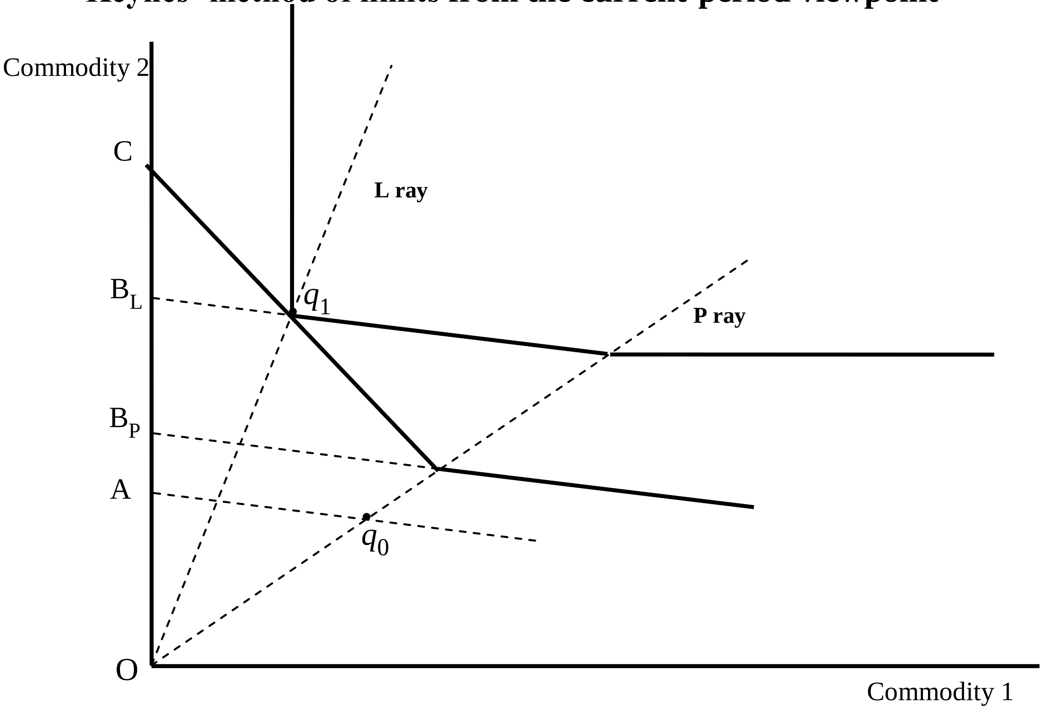


Figure 3: Laspeyres- and Paasche-type bounds based on q_1

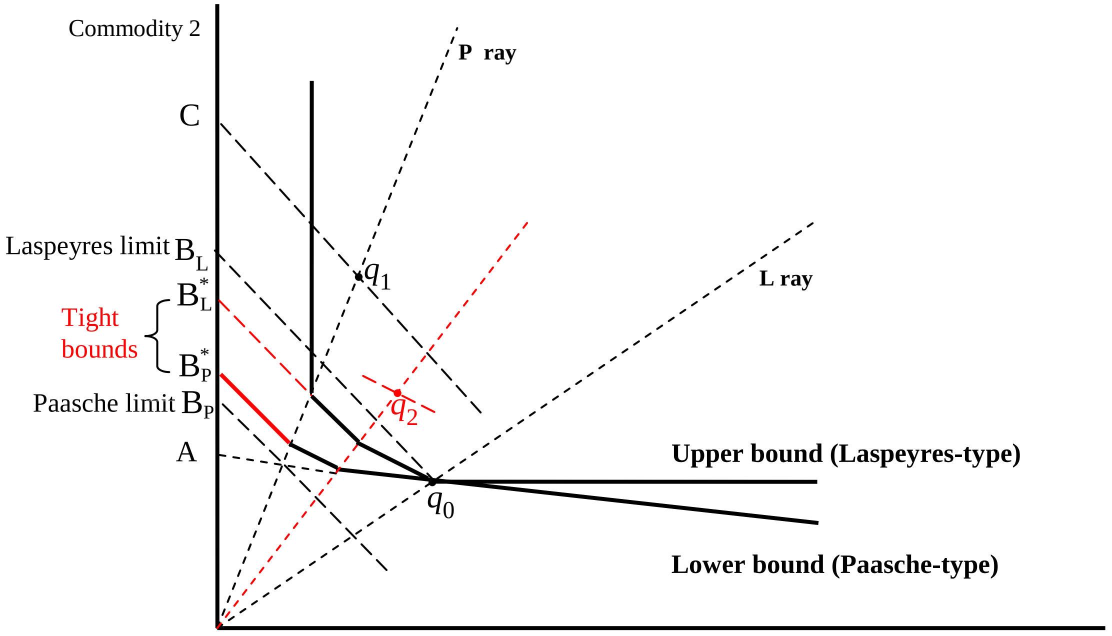
CA: Observed decrease in nominal expenditure (at the relative prices represented by the slope of A q_0 and C q_1, respectively);

CB_L: Price component of the decrease in nominal expenditure measured with the direct Laspeyres index number;

CB_P: Price component of the increase in nominal expenditure measured with the direct Paasche index number;

B_PA: Quantity component of the decrease in nominal expenditure measured with the implicit Paasche index number;

B_LA: Quantity component of the decrease in nominal expenditure measured with the implicit Laspeyres index number;



Samuelson-Afriat tight bounds

Figure 4: Tightening the bounds by adding a third observation point

A B_P^* : Price component of the increase in nominal expenditure measured with the chained Paasche index number (implicit chained Laspyeres index number);

A B_L^* : Price component of the increase in nominal expenditure measured with the chained Laspeyres index number (implicit chained Paasche index number).

Samuelson (1947) seems to be the first to recognize the advantage of a simultaneous use of all the data available in the construction of measures of aggregate of prices and quantities as shown in Figure 3. It is worth quoting Samuelson and Swamy's (1974) own words: "[…] Fisher missed the point made in Samuelson (1947, p. 151) that knowledge of a third situation can add information relevant to the comparison of two given situations. Thus Fisher contemplates Georgia, Egypt, and Norway, in which the last two each have the same price index relative to Georgia :

> "'We might conclude, since 'two things equal to the same thing are equal to each other,' that, therefore, the price levels of Egypt and Norway

> must equal, and this would be the case if we compare Egypt and Norway *via* Georgia. But, evidently, if we are intent on getting the very best comparison between Norway, we shall not go to Georgia for our weights … [which are], so to speak, none of Georgia's business.' [1922, p. 272].

"This simply throws away the transitivity of indifference and has been led astray by Fisher's unwarranted belief that only fixed-weights lead to the circular's test's being satisfied (an assertion contradicted by our P_i / P_j and Q_i / Q_j forms."

One of Afriat's main contribution in index number theory has been the development an original approach of constructing aggregating index numbers using all the data simultaneously (see Afriat, 1967, 1981, 1984, 2005). He also has developed an efficient algorithm to find the minimum path of chained upper limit index numbers (the Laspeyres indices on the demand side). In the following section this algorithm is briefly described.

4. The Afriat's power algorithm

With any chain of series of observation points $i, k, l, ..., m, j,$ there is the associated Laspeyres chain product $L_{ik} L_{kl} ... L_{mj}$. A simple chain among *n* elements is one without repeated elements, or loops. There are

$$n(n-1)...(n-r+1) = n!/r!$$

simple chains of length $r \le n$ and, therefore, altogether the finite number

$$n![1+1/1!+1/2!+...+1/(n-1)!]$$

Finding the optimal path efficiently among this number of possible chains becomes a computation problem as the number of elements increases. A way find this optimal path is to note that Hicks (1956)-Afriat (1977) LP inequality condition for the existence of the "true" aggregating index number is given by $K_{ij} \le L_{ij}$, that is the Paasche index does not exceed the Laspeyres index. This is is equivalent to the Afriat (1977) equivalent test

$$L_{ij} L_{ji} \ge 1 \tag{61}$$

Another way of stating this condition is that the 2×2 *L*-matrix

$$L \equiv \begin{bmatrix} 1 & L_{12} \\ L_{21} & 1 \end{bmatrix}$$

be idempotent, that is $L = L \cdot L$, in a modified arithmetic where + means *min.* In a series of theorems, Afriat has shown that also in the general case

$$M = (L)^n \quad \text{where} \quad n \geq 2 \tag{62}$$

with raising the $n \times n$-order matrix L of bilateral upper limit indexes to the power n is made in a modified arithmetic where + means *min.*

In a series of theorems, Afriat has shown that the existence of a solution

$$M_{ij} \equiv \min_{kl\ldots m} L_{ik} L_{kl} \ldots L_{mj} \qquad \text{where} \quad M_{ii} = 1 \tag{63}$$

is a necessary and sufficient test for reaching the minimum cycle satisfying the aggregation condition represented by the chained LP-inequality $H_{ij} \leq M_{ij}$ for every *i,j*.

A similar procedure can be applied to the lower bound indexes (the Paasche indexes on the demand side), where the matrix of these bilateral indexes is raised to the power *n* in a modified arithmetic where + means *max*. The solution

$$\begin{aligned} H_{ij} &\equiv \max_{kl\ldots m} K_{ik} K_{kl} \ldots K_{mj} \\ &= \max_{kl\ldots m} \frac{1}{L_{ki}} \frac{1}{L_{lk}} \ldots \frac{1}{L_{jm}} = \frac{1}{M_{ji}} \quad \text{where} \quad H_{ii} = 1 \end{aligned} \tag{64}$$

is also a necessary and sufficient simultaneous test for reaching the minimum cycle satisfying the aggregation condition, that is $H_{ij} \leq M_{ij}$ (the chained LP-inequality).

Diagonal elements $M_{ii} < 1$ and $H_{ii} < 1$ tell the inconsistency of the system. A critical efficiency parameter ε^* can be found for correction of the L matrix as described in the former section. For any element $M_{ii} < 1$, determine the number d_i of nodes in the path $i \ldots i$ and

$$e_i = (M_{ii})^{\frac{1}{d_i}} \tag{65}$$

If $M_{ii} \geq 1$, give e_i the value of 1 (that is $e_i = 1$) and then the critical efficiency parameter is determined as

$$\varepsilon^* = \min_i \varepsilon_i \tag{66}$$

The adjusted Laspeyres matrix is obtained as

$$L^{*} = L / \varepsilon^{*} \tag{67}$$

and the procedure goes on as before with L^{*} in place of the original *L*.

5. Multifactor productivity measurement

The methodology outlined above can be applied to the measurement of price and quantity aggregates in the analysis of production activities. The economic theory of production is isomorphic to the economic theory of consumption (see Samuelson, 1950, Samuelson and Swamy, 1974, p. 588, Muellbauer, 1971a, 1971b, Fisher and Shell, 1972, 1998, Diewert, 1983, and Fisher, 1995). In the theory of output supply, the only difference with the theory of input demand is that convexity leads to maximization where concavity leads to minimization and all bounds are reversed.

An invariant index numbers of output, input, and productivity can be constructed if the necessary and sufficient chain-consistent Laspeyres-Paasche inequality condition is satisfied. A data set giving account of all outputs and inputs is most likely to be consistent with this condition. A convenient way is to consider the accounting equations of nominal value of net profits:

$$\Pi_t = p_{yt} y_t - w_t x_t \tag{68}$$

where p_y and *y* are the price and quantities of outputs and *w* and *x* are the prices and quantities of inputs. If the set of outputs and inputs is complete, we might expect that the conditions of aggregation of input and output quantities are fulfilled. In this case the CLP-inequality condition should be satisfied. Following Samuelson (1950, p. 23) and Debreu (1959, p. 38), we might consider inputs as negative outputs and convert the foregoing accounting equation in the form

$$\Pi_t = p_t q_t \tag{69}$$

where $p \equiv [\, p_y \;\; w]$ and $q \equiv [\, y \;\; \text{-}x]$. By applying the definitions and methodology outlined in the previous section, we could calculate the bounds of the price and quantity indexes between two situations:

$$\Pi_{ij} = \overset{\frown}{P}_{ij} \breve{Q}_{ij} = \breve{P}_{ij} \overset{\frown}{Q}_{ij} \tag{70}$$

where

$$\Pi_{ij} \equiv \frac{\Pi_i}{\Pi_j}\,, \quad \overset{\frown}{P}_{ij} \equiv \frac{\overset{\frown}{P}_i}{\overset{\frown}{P}_j}\,, \quad \breve{Q}_{ij} \equiv \frac{\breve{Q}_i}{\breve{Q}_j}\,, \quad \breve{P}_{ij} \equiv \frac{\breve{P}_i}{\breve{P}_j}\,, \quad \overset{\frown}{Q}_{ij} \equiv \frac{\overset{\frown}{Q}_i}{\overset{\frown}{Q}_j}$$

with $\overset{\frown}{P}_{ij}$, $\breve{P}_{ij}$, $\overset{\frown}{Q}_{ij}$, and $\breve{Q}_{ij}$ have been defined in the previous section as the tight chain-consistent upper and lower bounds of prices and quantities, respectively. As already stated, for *outputs* the upper bounds are given by Paasche index numbers, whereas the lower bounds are given by Laspeyres index numbers, that is

$$\overset{\frown}{P}_{ij} \geq \breve{P}_{ij} \tag{71}$$

$$\overset{\frown}{Q}_{ij} \geq \breve{Q}_{ij} \tag{72}$$

If these conditions are fulfilled, we could apply the Shephard-Afriat's factorization theorem and redefine

$$\Pi_t = P(p_t).Q(q_t) \tag{73}$$

where $P(p)$ and $Q(q)$ are the "true" aggregator functions of prices and quantities respectively. The chain-consistent index numbers could be calculated by following the methodology outlined in the previous section.

On difficulty in constructing price and quantity indexes of net profits is that in the long run equilibrium in competitive markets, there have null numerical values. Any index number defined as a ratio where the null net profits enter in the denominator become indeterminate. This problem has made the economic measurements of productivity based on decomposition of net profits a "road less travelled" in the empirical literature (the few exceptions include Archibald, 1977, Balk, 1998 and Diewert, 2000). However, as it has been stressed in Färe and Primont (1995, p. 149) and Milana (2006), the accounting for net profits may turn the analytical framework much wider than that based on the analysis of revenues (gross profits) and costs of production. Among other things, this could allow us to take into account of possible "hidden" inputs or outputs, which may be the cause of non-constant returns to scale and non-zero net profits.

The analysis of changes in net profits could be carried out in terms of first differences rather than ratios (see Diewert, 2000, 2005 for a previous definition). If we normalize the net profits by the nominal value of one output, say output 1, we get normalized net profits. The first differences of the values taken by these normalized net profits in two observation points *i* and *j* can be decomposed as follows

$$\tilde{\Pi}_i - \tilde{\Pi}_j = \underset{\substack{\text{Paasche-type}\\ \text{quantity}\\ \text{component}}}{\tilde{p}_i(\tilde{q}_i - \tilde{q}_j)} + \underset{\substack{\text{Laspeyres-type}\\ \text{price}\\ \text{component}}}{(\tilde{p}_i - \tilde{p}_j)\tilde{q}_j} \tag{74}$$

or, alternatively,

$$\tilde{\Pi}_i - \tilde{\Pi}_j = \underset{\substack{\text{Laspeyres-type}\\ \text{quantity}\\ \text{component}}}{\tilde{p}_j(\tilde{q}_i - \tilde{q}_j)} + \underset{\substack{\text{Paasche-type}\\ \text{price}\\ \text{component}}}{(\tilde{p}_i - \tilde{p}_j)\tilde{q}_i} \qquad (75)$$

where $\tilde{\Pi}_t \equiv \Pi_t / p_{t1}q_{t1}$, $P(\tilde{p}_t) \equiv P(p_t)/p_{t1}$, with $\tilde{p}_t \equiv p_t / p_{t1}$ $Q(\tilde{q}_t) = Q(q_t)/q_t$ with $\tilde{q}_t \equiv q_t / q_{t1}$.

The following remarks can be made regarding the foregoing decomposition procedures[11]:

Remark no. 1: The two alternative decompositions offer upper and lower bounds in the typical direct bilateral comparison.

Remark no. 2: The direct bilateral comparisons do not satisfy, in general, the circularity test of the price and quantity components (except in the very special case where the data are consistent with given fixed weights). We may apply the optimized chaining procedure outlined in the previous section in order to define chain consistent tight bounds.

Remark no. 3: Using normalized prices and quantities, the price component has the meaning of relative multifactor productivity change *distributed* among changes in real rewards of factor inputs, whereas the quantity component has the meaning of relative change in multifactor productivity *originated* from changes in average productivity of factor inputs .

A very simple example will illustrate Remark no. 3. In a stylized model of one output (*y*) and one input (*x*) of a producer facing competitive markets, the Paasche-type quantity component defined above becomes

$$\tilde{p}_i(\tilde{q}_i - \tilde{q}_j) = \frac{w_i}{p_i}\left(1 - \frac{x_i}{y_i}\right) - \frac{w_i}{p_i}\left(1 - \frac{x_j}{y_j}\right)$$

$$= \frac{w_i}{p_i}\left(-\frac{x_i}{y_i} + \frac{x_j}{y_j}\right) = \frac{w_i}{p_i} \cdot \frac{y_i x_j - y_j x_i}{y_j y_i}$$

---

[11] Diewert (2000)(2005) considers the Bennet-type decomposition constructed as an arithmetic mean of these two alternative decompositions. For the purposes of our methodology, we shall disregard this third procedure.

$$= \frac{w_i}{p_i} / \frac{y_j}{x_i} \cdot \frac{y_i x_j - y_j x_i}{y_j x_i} \cdot$$

$$= \frac{y_i x_j - y_j x_i}{y_j x_i}, \text{ since, in a competitive equilibrium, } \frac{w_i}{p_i} = \frac{y_i}{x_i}$$

$$= \left(\frac{y_i}{x_i} - \frac{y_j}{x_j} \right) / \frac{y_j}{x_j} \longleftarrow \text{Laspeyres-type rate of change in productivity} \qquad (76)$$

Similarly, by defining a Laspeyres-type technical change component, we obtain $\tilde{p}_j(\tilde{q}_i - \tilde{q}_j) = \left(1 - \frac{x_i}{y_i}\frac{w_j}{p_j}\right) - \left(1 - \frac{x_j}{y_j}\frac{w_j}{p_j}\right) = \frac{y_i x_j - y_j x_i}{y_i x_j}$, which corresponds to a Paasche-type rate of change in productivity. This result leads us to the useful result that, starting from profit accounts, we can calculate the index number of productivity by simply adding 1 to the foregoing formula, that is

$$\frac{y_i}{x_i} / \frac{y_j}{x_j} = \tilde{p}_i(\tilde{q}_i - \tilde{q}_j) + 1 \qquad (77)$$

6. An application to EU KLEMS data

The method of chain-consistent index numbers is ideally designed for multilateral comparisons both at an intertemporal and interspatial context. It seems to be the only index number method that fulfils the circularity requirement and other important conditions for economic index numbers. The EU KLEMS database on productivity and economic growth at industry level recently constructed by a consortium of institutions of more than 16 countries in a EU funded research project is particularly suitable for the application of this method[12].

Interspatial comparisons require, in particular, information concerning relative levels of prices, the Purchasing Power Parities (PPP) and/or quantities across the examined geographical units. This type of data is going to be an integral part of the EU KLEMS project at industry level, covering more than 30 countries. At the current stage of the construction, however, this information is only provisional and, therefore, we have resolved to use and update the PPP data constructed in our previous works (see, for example, Milana, 2001 and Fujikawa and Milana, 1996a, 1996b).

We consider a subset of 10 countries (the US, Japan, the six founding member countries of the EEC, the UK and Spain), for which price and quantities of outputs and inputs are available at disaggregated level for the period 1995-2005. In the EU

---

[12] The EU KLEMS database is available for free download on the web site www.euklems.net .

KLEMS estimates, during this period, multifactor productivity in the overall economy appears to have grown faster in the US, by 13 per cent, followed by the other countries at much slower speed (France with 6.2 per cent, Germany and Netherlands around 4-6-4.7 per cent, UK with 3.2 per cent, Japan with 2.7 per cent), whereas it has even decreased in others (Belgium and Luxembourg with -2.6-2.7 per cent, Italy with -4.5 per cent, and Spain with -7.5 per cent).

However, these estimates are difficult to compare both directly on a bilateral basis and indirectly in a multilateral context. They are constructed using procedures of aggregation of outputs and inputs across the industries that hardly satisfy the required consistency criteria. As noted elsewhere (see for example Milana, 2006), the LP-inequality test often fails. The obtained estimates do not have, in this case, the legitimate meaning of aggregates. Moreover, the use of a specific index number formula adds further difficulties since the aggregation is jointly tested with the chosen functional form.

In the general case, the chosen index number formula does not only fail to satisfy the transitivity requirement, but also affects the ranking of the countries' position. Figures 4 and 5 show the relative levels of MFP with respect to that in the US in 1995 along a different path of chaining the upper limit (Paasche) indexes. The ranking position of the countries changes according to the chosen path. This unwarrant result has been already noted in previous empirical studies.

The price level solution allows us to construct the matrix of chain-consistent "true" index numbers. Tables 3 and 4 show the resulting transitive index numbers of multifactor productivity levels with respect to that in the US in 1995 and 2005, respectively. These are obtained by chosing the highest and the lowest levels of MFP levels resulting, *mutatis mutandis*, from the application of formulas (59) and (60). These level estimates have been obtained after an adjustment for inefficiency. The parameters of adjustments are given below the tables. Part of the gap in MFP levels with respect to the reference country appears to be due to inefficiency.

The country ranking seems to be less influenced although some changes are worth noting. For example, Belgium now appears below Italy and the Netherlands below France, whereas they were in a reversed position in the original comparison.

We should emphasize, however, that these results are only provisional. A number of refinements should be made on the data. In particular, the PPP data considered here should be carefully scrutinized also in the light of the alternative estimates that will hopefully become available in the immediate future. More countries should be added to the set of comparisons in order to acquire further information on the technology in use.

8. Conclusive remarks

Multilateral index numbers have always been considered as problematic since Irving Fisher's (1922) rebuttal of the property of circularity. Since then, transitive measures have been achieved either by “smoothing out” inconsistencies of bilateral comparisons, or by extending the index number approach to some parametric estimation of the underlying economic functions, thus resolving to simulations in order to take into account price-induced substitution effects.

All the difficulties of constructing economic index numbers that have been identified under the various formulations of an impossibility theorem can be overcome by relinquishing a point estimate and accepting to deal with a (tightly) bounded closed interval of alternative candidates of the unknown “true” measures. However, the tighter the bounds the more useful these are for potential users of the estimates.

The method proposed does not require that a non-observational object like a utility or technological function governing the economic agents’ behaviour really exists. Rather, it only finds it convenient to test whether the observed data can be rationalized with such theoretical functions. Assuming that these can be of general forms including those that are multivalued (linear peace-wise), similar to the functions used in activity analysis, we can try to find optimized chained Laspeyres and Paasche index numbers. These should be “exact” for such functions and can be considered as tight bounds of the interval of numerical values comprising the unknown “true” measure. If these functions fit the observed data perfectly, then these data may have been generated by rational behaviour and no inefficiency has occured.

A number of difficulties, however, may arise when applying the method to the actual data. Many of these may arise from failures of the LP-inequality test in some bilateral comparisons. This does not necessarily entails inefficiency *per se*. The LP-inequality condition for the existence of an aggregating index are very clearly stated in the demand or supply contexts, but may become somewhat indeterminate at a macro level, where demand and supply interact with each other thus offsetting totally or partially their opposite directions when price conditions tend to change. This is a classical difficulty in index number theory, which arises when aggregation takes place at various levels simultaneously (over outputs, over firms, and over all industries and consumers).

Moreover, even in a partial equilibrium context, the solutions found in the usual situation of inefficiency may not be univocal not only in terms of relative levels, but also in their ranking. This is because the aggregation refers to an underlying utility or technology function that does not fit the data perfectly, but only approximately. Despite all these problems, the methodology applied here appears to respond positively to the long-standing search for a definite answer to the “index number problem”.

Table 2. Starting matrix of (upper-limit) Paasche index numbers of TFP in all industries in 1995 and 2005 (USA = 1)*

1995

	USA	Jap	Ger	Fra	Ita	Bel	Nld	Lux	UK	Spa
USA	1.00000	1.17086	1.15854	1.16955	1.17269	1.19603	1.15829	1.16175	1.17625	1.22387
Japan	0.84719	1.00000	0.99013	1.02000	1.11651	1.11965	1.02401	1.13456	1.04555	1.12964
Germany	0.86099	1.01567	1.00000	1.05612	1.12098	1.11515	1.05230	1.14431	1.09703	1.14792
France	0.82952	0.98496	0.95437	1.00000	1.06410	1.04251	0.99178	1.07340	1.01351	1.08395
Italy	0.81380	0.91370	0.90322	0.94568	1.00000	0.98414	0.95685	1.00752	0.96015	1.00579
Belgium	0.79836	0.91330	0.90933	0.96544	1.02518	1.00000	0.94256	1.01446	0.97846	1.08180
Netherl	0.83921	0.98124	0.95914	1.01054	1.07439	1.06921	1.00000	1.07247	1.02459	1.10858
Luxemb	0.82883	0.89235	0.88547	0.93934	0.99443	0.98925	0.94004	1.00000	0.95013	1.00057
UK	0.81842	0.96140	0.92137	0.98934	1.05443	1.02925	0.98004	1.07000	1.00000	1.05950
Spain	0.77212	0.89792	0.88325	0.93490	0.99547	0.93481	0.91560	0.99956	0.95577	1.00000

2005

	USA	Jap	Ger	Fra	Ita	Bel	Nld	Lux	UK	Spa
USA	1.00000	1.28828	1.25038	1.24444	1.38758	1.38760	1.25131	1.34368	1.28795	1.49511
Japan	0.76997	1.00000	0.97121	0.98639	1.20068	1.18057	1.00541	1.19263	1.04048	1.25421
Germany	0.79775	1.03545	1.00000	1.04121	1.22896	1.19873	1.05331	1.22630	1.11297	1.29932
France	0.77960	1.01853	0.96804	1.00000	1.18332	1.13670	1.00695	1.16679	1.04297	1.24450
Italy	0.68777	0.84964	0.82385	0.85040	1.00000	0.96494	0.87361	0.98483	0.88851	1.03841
Belgium	0.68815	0.86617	0.84593	0.88544	1.04558	1.00000	0.87768	1.01134	0.92346	1.13911
Netherl	0.77682	0.99939	0.95822	0.99532	1.17677	1.14825	1.00000	1.14821	1.03849	1.25360
Luxemb	0.71661	0.84891	0.82627	0.86416	1.01734	0.99230	0.87803	1.00000	0.89950	1.05681
UK	0.74744	0.96608	0.90817	0.96139	1.13945	1.09054	0.96692	1.13024	1.00000	1.18206
Spain	0.63205	0.80874	0.78033	0.81430	0.96420	0.88778	0.80968	0.94636	0.85667	1.00000

*** Comparison countries are shown by rows, base countries are shown by columns.**
Source: Our computation on EU KLEMS database.

Table 3. Chain-consistent “true” MFP relative levels in all industries in 1995 (USA = 1)*

Ratios of upper true measures of relative levels:

	USA	Jap	Ger	Fra	Ita	Bel	Nld	Lux	UK	Spa
USA	1.00000	1.09084	1.07335	1.11407	1.15308	1.15756	1.10121	1.25711	1.12918	1.19690
Japan	0.91673	1.00000	0.98397	1.02130	1.05706	1.06116	1.00951	1.15242	1.03515	1.09723
Germany	0.93166	1.01629	1.00000	1.03794	1.07428	1.07845	1.02595	1.17119	1.05201	1.11510
France	0.89761	0.97914	0.96345	1.00000	1.03501	1.03903	0.98845	1.12839	1.01356	1.07434
Italy	0.86724	0.94602	0.93086	0.96617	1.00000	1.00388	0.95502	1.09022	0.97928	1.03800
Belgium	0.86389	0.94236	0.92726	0.96244	0.99613	1.00000	0.95132	1.08600	0.97549	1.03398
Netherl.	0.90809	0.99058	0.97470	1.01168	1.04710	1.05117	1.00000	1.14157	1.02540	1.08689
Luxemb.	0.79548	0.86774	0.85383	0.88622	0.91725	0.92081	0.87599	1.00000	0.89824	0.95210
UK	0.88560	0.96604	0.95056	0.98662	1.02116	1.02513	0.97523	1.11329	1.00000	1.05996
Spain	0.83549	0.91139	0.89678	0.93080	0.96339	0.96713	0.92006	1.05031	0.94343	1.00000

Ratios of lower true measures of relative levels:

	USA	Jap	Ger	Fra	Ita	Bel	Nld	Lux	UK	Spa
USA	1.00000	1.09084	1.16872	1.20398	1.26894	1.29420	1.20871	1.25711	1.23414	1.32432
Japan	0.91673	1.00000	1.07140	1.10372	1.16327	1.18643	1.10806	1.15242	1.13137	1.21404
German	0.85564	0.93336	1.00000	1.03017	1.08575	1.10736	1.03422	1.07562	1.05597	1.13314
France	0.83058	0.90603	0.97072	1.00000	1.05396	1.07493	1.00393	1.04412	1.02505	1.09996
Italy	0.78806	0.85964	0.92102	0.94881	1.00000	1.01990	0.95254	0.99067	0.97257	1.04364
Belgiu	0.77268	0.84287	0.90305	0.93029	0.98049	1.00000	0.93395	0.97134	0.95359	1.02328
Nether	0.82733	0.90248	0.96691	0.99608	1.04983	1.07072	1.00000	1.04004	1.02103	1.09565
Luxemb	0.79548	0.86774	0.92969	0.95774	1.00942	1.02951	0.96151	1.00000	0.98173	1.05347
UK	0.81028	0.88389	0.94699	0.97556	1.02820	1.04867	0.97940	1.01861	1.00000	1.07308
Spain	0.75510	0.82369	0.88250	0.90913	0.95818	0.97725	0.91270	0.94924	0.93190	1.00000

* Comparison countries are shown by rows, base countries are shown by columns.
Source: Our computation on EU KLEMS database.

Vector *e*: [0.993059 0.993059 0.995151 0.978530 0.962345 1.000000 0.976827 0.993059 0.979559 0.996915].

e^*: 0.962345.

Table 4. Chain-consistent "true" MFP relative levels in all industries in 2005 (USA = 1)*

Ratios of upper true measures of relative levels:

	USA	Jap	Ger	Fra	Ita	Bel	Nld	Lux	UK	Spa
1 USA	1.00000	1.20024	1.15844	1.18541	1.47714	1.34295	1.18966	1.36393	1.23642	1.46215
2 Japan	0.83317	1.00000	0.96518	0.98765	1.23070	1.11890	0.99118	1.13638	1.03014	1.21821
3 German	0.86323	1.03608	1.00000	1.02328	1.27511	1.15927	1.02694	1.17738	1.06731	1.26216
4 France	0.84359	1.01251	0.97725	1.00000	1.24609	1.13289	1.00358	1.15059	1.04303	1.23345
5 Italy	0.67698	0.81254	0.78425	0.80251	1.00000	0.90915	0.80538	0.92336	0.83704	0.98985
6 Belgiu	0.74463	0.89374	0.86261	0.88270	1.09992	1.00000	0.88586	1.01562	0.92068	1.08876
7 Nether	0.84058	1.00890	0.97376	0.99643	1.24165	1.12885	1.00000	1.14649	1.03931	1.22905
8 Luxemb	0.73318	0.87999	0.84934	0.86912	1.08300	0.98462	0.87223	1.00000	0.90651	1.07201
9 UK	0.80879	0.97074	0.93694	0.95875	1.19469	1.08616	0.96218	1.10313	1.00000	1.18256
10 Spain	0.68393	0.82088	0.79229	0.81074	1.01025	0.91848	0.81364	0.93283	0.84562	1.00000

Ratios of lower true measures of relative levels:

	USA	Jap	Ger	Fra	Ita	Bel	Nld	Lux	UK	Spa
1 USA	1.00000	1.20706	1.26136	1.28108	1.50147	1.50149	1.30578	1.45396	1.35133	1.61782
2 Japan	0.82846	1.00000	1.04498	1.06132	1.24390	1.24392	1.08178	1.20455	1.11951	1.34030
3 German	0.79279	0.95695	1.00000	1.01563	1.19036	1.19037	1.03521	1.15269	1.07132	1.28260
4 France	0.78059	0.94223	0.98461	1.00000	1.17204	1.17205	1.01928	1.13496	1.05484	1.26286
5 Italy	0.66602	0.80392	0.84009	0.85322	1.00000	1.00001	0.86967	0.96836	0.90000	1.07749
6 Belgiu	0.66601	0.80391	0.84007	0.85320	0.99999	1.00000	0.86966	0.96835	0.89999	1.07748
7 Nether	0.76583	0.92440	0.96598	0.98108	1.14986	1.14988	1.00000	1.11348	1.03488	1.23897
8 Luxemb	0.68777	0.83019	0.86753	0.88109	1.03267	1.03269	0.89808	1.00000	0.92941	1.11270
9 UK	0.74001	0.89324	0.93342	0.94801	1.11111	1.11112	0.96629	1.07595	1.00000	1.19721
10 Spain	0.61811	0.74610	0.77967	0.79185	0.92808	0.92809	0.80712	0.89872	0.83527	1.00000

---

* Comparison countries are shown by rows, base countries are shown by columns.
Source: Our computation on EU KLEMS database

Vector e: [0.971715 0.981053 0.985539 0.978531 0.962028 0.985756 0.976827 0.959626 0.979560 0.981907]

e^*: 0.959626.

Figure 4
Relative position in MFP relative levels in 1995 (USA = 1)
(Estimates with upper limit index numbers)

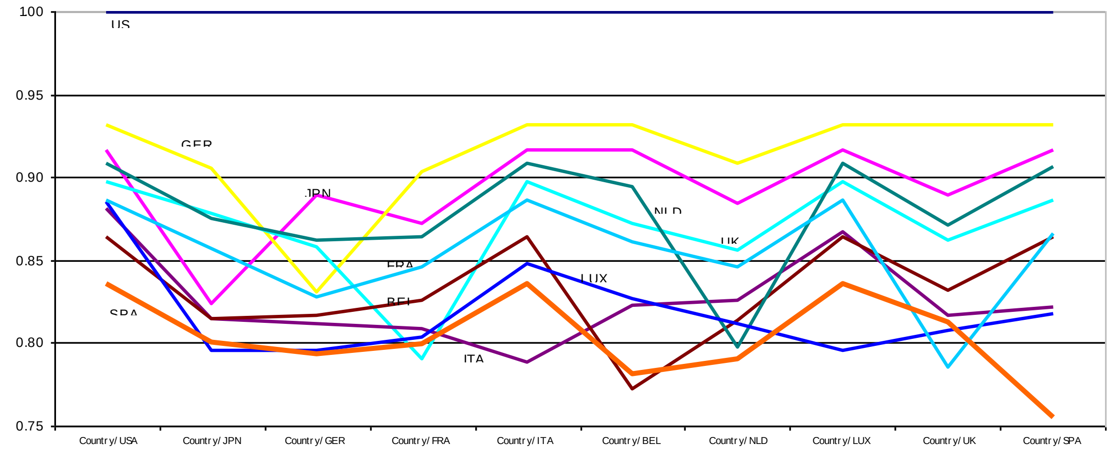


Figure 5
Relative position in MFP relative levels in 1995 (USA=1)
(Estimates with lower limit index numbers)

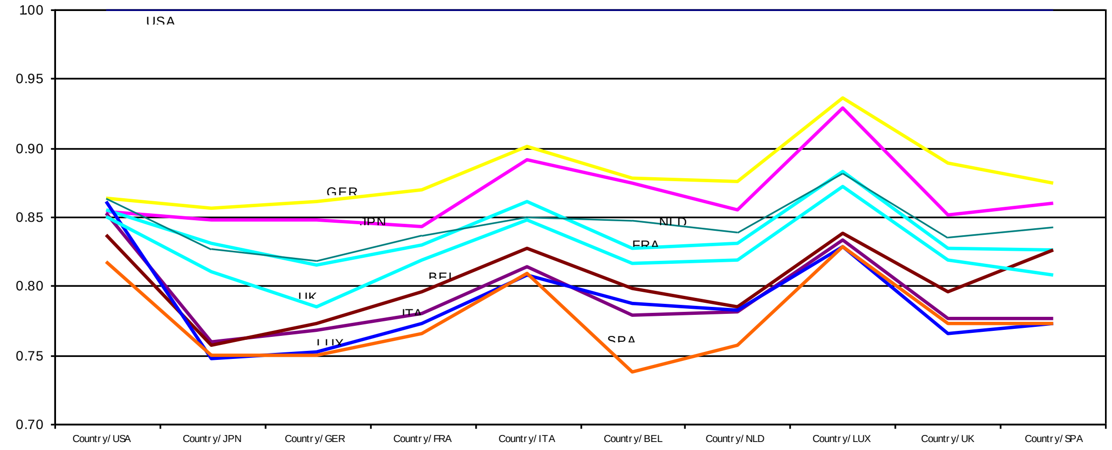


Figure 6
Upper and lower bounds of TFP relative levels in 1995 (USA = 1)

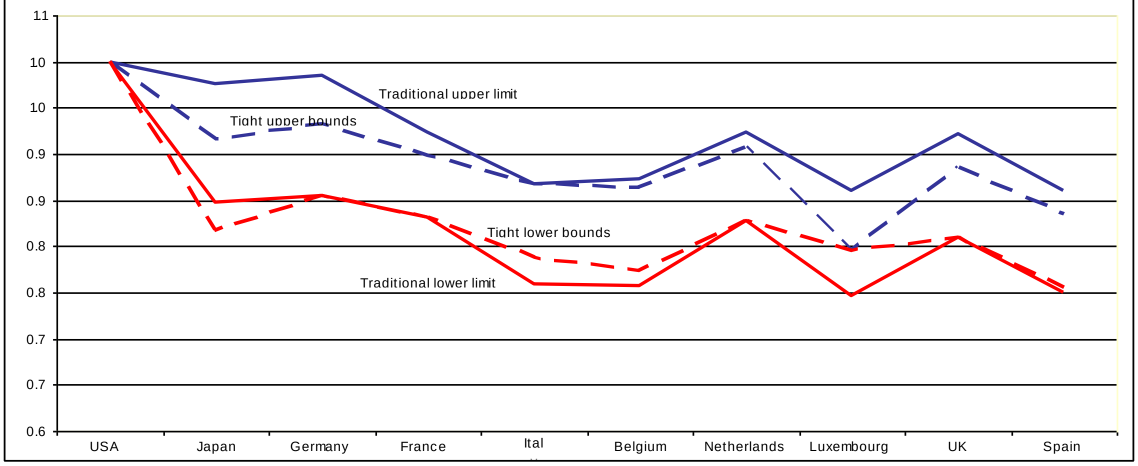
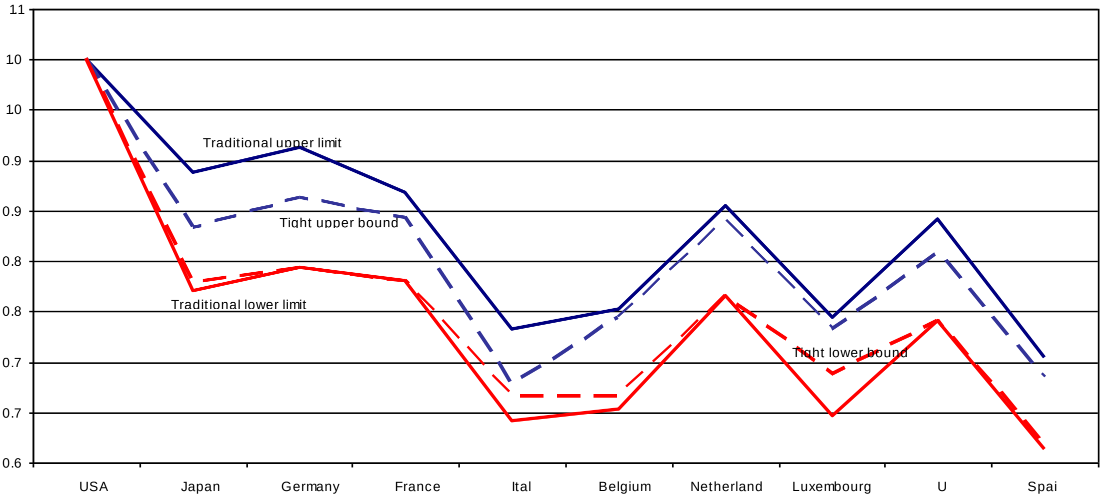


Figure 7
Upper and lower bounds of TFP relative levels in 2005 (USA =1)



References

Afriat, S. N. (1953-56). Theory of Economic Index Numbers. The Consistency Condition and Other Concepts in the Theory of Value and Demand. The Calculation of Index Numbers of the Standard and Cost of Living. On Index Numbers in the Theory of Value and Demand. *Research Reports* (mimeo.), Department of Applied Economics, Cambridge.

Afriat, S. N. (1959). Value and Expenditure. *Research Memorandum* No. 7 (February), Econometric Research Program, Princeton University.

Afriat, S. N. (1967a). The Cost of Living Index. In Martin Shubik (ed.), *Studies in Mathematical Economics in Honor of Oscar Morgenstern.* Princeton University Press.

Afriat, S. N. (1967b), "The Construction of Utility Functions from Expenditure Data", *International Economic Review* 8: 67-77.

Afriat, S. N. (1971a). The Output Limit Function in General and Convex Programming and the Theory of Production. *Econometrica* 39, 309-39.

Afriat, S.N. (1972), "The Theory of International Comparisons of Real Income and Prices", In In D. J. Daly (ed. by), *International Comparisons of Price and Output*, Studies in Income & Wealth, No.37, National Bureau of Economic Research, New York: Columbia University Press, pp. 208-216.

Afriat, S.N. (1977), *The Price Index*, Cambridge, UK: Cambridge University Press.

Afriat, S. N. (1981), "On the Constructability of Consistent Price Indices Between Several Periods Simultaneously", in Angus Deaton (ed. by), *Essays in the Theory and Measurement of Consumer Behaviour: in Honour of Sir Richard Stone*, Cambridge, UK: Cambridge University Press, 133-161.

Afriat, S. N. (1984), "The True Index", in A. Ingham and A.M. Ulph (eds.), *Demand, Equilibrium and Trade: Essays in Honor of Ivor F. Pearce*. London: Macmillan, 37-56. Reproduced in S. N. Afriat, *The Price Index and Its Extension: A chapter in economic measurement*, London and New York: Routledge, 2005, pp. 157-173.

Afriat, S.N. (2005). *The Price Index and Its Extension. A Chapter in Economic Measurement*. London and New York: Routledge.

Afriat, S. N. and Nuri T. Jazairi (1988). Fisher's Test Approach to Index Numbers. In Samuel Kotz and Norman L. Johnson (eds.), *Encyclopedia of Statistical Sciences*. New York: John Wiley & Sons.

Afriat, S.N. and Milana (2008), *Economics and the Price Index*. London and New York: Routledge.

Allen, Robert C. and Erwin W. Diewert (1981), “Direct Versus Implicit Superlative Indexe Number Formulae”, *Review of Economics and Statistics* 63: 430-435.

Antonelli, G.B. (1886), *Sulla teoria matematica dell’economia pura.* Reprinted in *Giornale degli Economisti* 10 (1951): 233-263.

Archibald, R. (1975), “On the Theory of Industrial Price Measurement: Input Price Indexes”, *Annals of Economic and Social Measurement* 6: 57-72.

Balk, Bert M. (1998), *Industrial Price, Quantity, and Productivity Indices*. *The Micro-Economic Theory and An Application*, Boston, Dordrecht, London, Kluwer Academic Publishers.

Blundell, Richard (2005). How Revealing is Revealed Preference? *European Economic Journal* 3, 211-35.

Blundell, Richard W., Martin Browning and Ian A. Crawford (2003). Nonparametric Engel Curves and Revealed Preferences. *Econometrica* 71, 208-40.

Caves, Douglas W.; Laurits R. Christensen; and W. Erwin Diewert (1982a), "The Economic Theory of Index Numbers and the Measurement of Input, Output, and Productivity, *Econometrica* 50: 1393-1414.

Caves, Douglas W.; Laurits R. Christensen; and W. Erwin Diewert (1982b), “Multilateral Comparisons of Output, Input, and Productivity Using Superlative Index Numbers”, *Economic Journal* 92: 73-86.

Chavas, Jean-Paul and Thomas L. Cox (1990), “A Non-Parametric Analysis of Productivity: The Case of U.S. and Japanese Manufacturing”, *American Economic Review* 803, 450-64.

Chavas, J.-P. and T. Cox (1992). A Nonparametric Analysis of the Influence of Research on Agricultural Productivity, *American Journal of Agricultural Economics*, 74(3), 583-91.

Chavas, J.-P. and T. Cox (1994). A Primal-Dual Approach to Nonparametric Productivity Analysis: The Case of U.S. Agriculture. *Journal of Productivity Analysis* 5, 359-73.

Chavas, J.-P., M. Aliber and T. I. Cox (1994). A Nonparametric Analysis of the Source and Nature of Technical Change: The Case of U.S. Agriculture. University of Wisconsin-Madison, Department of Agricultural Economics, *Staff Paper* Series No. 373.

Cox, Thomas L. and Chavas, Jean-Paul (1990), “A Nonparametric Analysis of Productivity: The Case of US Agriculture”, *European Review of Agricultural Economics* 17: 449-64.

Deaton, Angus (1986), “Demand Analysis”, in Z. Griliches and M.D. Intriligator (eds.), *Handbook of Econometrics,* Vol. III, Elsevier Science Publishers BV, New York, pp. 1767-1839.

Denny, Michael (1972), “Trade and Production Sector: An Exploration of Models of Multi-Product Technology”, Ph. D. Dissertation, University of California at Berkeley.

Denny, Michael (1974), "The Relationship Between Functional Forms for the Production System", *Canadian Journal of Economics* 7: 21-31.

Diewert, W. Erwin (1971), "An Application of the Shephard Duality Theorem: A Generalized Leontief Production Function", Journal of Political Economy 79: 481-507.

Diewert, W. Erwin (1976), "Exact and Superlative Index Numbers", *Journal of Econometrics* 4: 115-145.

Diewert, W. Erwin (1973), “Afriat and Revealed Preference Theory”, *Review of Economic Studies* 40: 419-425.

Diewert, W. Erwin (1980), “Aggregation Problems in the Measurement of Capital”, in D. Usher (ed. by), *The Measurement of Capital*, Studies in Income and Wealth, Vol. 45, National Bureau of Economic Research, Chicago, University of Chicago Press, pp. 433-528.

Diewert, W. Erwin and Celik Parkan (1983), “Linear Programming Tests of Regularity Conditions for Production Functions”, in *Quantitative Studies on Production and Prices.* Würzburg-Wien: Physica-Verlag, 131-58.

Diewert, W. Erwin and Celik Parkan (1985), “Tests for Consistency of Consumer Data”, *Journal of Econometrics* 30 (1-2), 127-47.

Diewert, W. Erwin (2000), “Productivity Measurement Using Differences rather than Ratios: A Note”, Discussion Paper 2000/1, University of New South Wales, School of Economics

Diewert, W. Erwin (2005), “Index Number Theory Using Differences Rather Than Ratios,” University of British Columbia, Department of Economics, *American Journal of Economics and Sociology* 64: 311-360.

Diewert, W. Erwin and Alice O. Nakamura (2003), “Index Number Concepts, Measures and Decomposition of Productivity Growth”, *Journal of Productivity Analysis* 19: 127-159.

Dowrick, Steve and John Quiggin (1994). International Comparisons of Living Standards and Tastes: A Revealed-Preference Analysis. *American Economic Review* 84, 332-41.

Dowrick, Steve and John Quiggin (1997). True Measures of GDP and Convergence. *American Economic Review* 87 (1), 41-64.

Famulari, Melissa (1995), A Household-Based, Nonparametric Test of Demand Theory. *Review of Economics and Statistics* 372-82.

Färe, Rolf and Daniel Primont (1995), *Multi-Output Production and Duality: Theory and Applications*, Norwell, MA, Kluwer Academic Publishers.

Fisher, Franklin M. and Karl Shell (1972), *Economic Theory of Price Indices*, New York: Academic Press.

Fisher, Franklin M. (1995), “The Production-Theoretic Measurement of Input Price and Quantity Indices”, *Journal of Econometrics 65*: 155-174.

Fisher, Franklin M. (1988), "Production-Theoretic Input Price Indices and the Measurement of Real Aggregate Input Use", in W. Eichhorn (ed. by), *Measurement in Economics*, Heidelberg, Physica-Verlag, pp. 87-98.

Fisher, Franklin M. and Karl Shell (1998), *Economic Analysis of Production Price Indexes*, Cambridge, UK: Cambridge University Press.

Frisch, Ragnar (1930), “Necessary and Suffient Conditions Regarding the Form of an Index Number Which Shall Meet Certain of Fisher’s Tests,” *Journal of American Statistical Association* 25: 297-406.

Frisch, Ragnar (1936), “Annual Survey of General Economic Theory: The Problem of Index Numbers”, *Econometrica* 4: 1-39.

Fujikawa, Kiyoshi and Carlo Milana (1996a), “Bilateral and Multilateral Comparisons of Productivity in Input-Output Analysis Using Alternative Index Numbers”, in OECD, *Industry, productivity. International Comparisons and Measurement Issues*. Paris, pp. 217-241.

Fujikawa, Kiyoshi and Carlo Milana (1996b), “Direct and Indirect Components of Producer Prices in International Comparisons”, Presented at the 1996 Conference of Pan Pacific Association of Input-Output Studies held at the University of Nagoya, Nagoya, Japan on 9th-10th November, 1996.

Gorman, W.M. (1953), "Community Preference Fields", *Econometrica* 21: 63-80.

Gorman, W.M. (1961), "On a Class of Preference Fields", *Metroeconomica* 13: 53-56.

Hanoch, Giora and Michael Rothschild (1972), "Testing the Assumptions of Production Theory: A Nonparametric Approach", *Journal of Political Economy* 80 (2), 256-75.

Hicks, J.R. (1956), *A Revision of Demand Theory*. Oxford: The Clarendon Press.

Hill, Robert J. (1997), "A Taxonomy of Multilateral Methods for Making International Comparisons of Prices and Quantities", *Review of Income and Wealth* 43(1): 49-69.

Hill, Robert J. (1999a), "Comparing Price Levels across Countries Using Minimum-Spanning Trees", *Review of Economics and Statistics* 81(1): 135-142.

Hill, Robert J. (1999b), "International Comparisons Using Spanning Trees", in A. heston and R.E. Lipsey (eds.), *International and Interarea Comparisons of Income, Output, and Prices,* Studies in Income and Wealth, Volume 61, NBER, Chicago: The University of Chicago Press.

Hill, Robert J. (2001), "Linking Countries and Regions Using Chaining Methods and Spanning Trees", prepared for the Joint World Bank-OECD Seminar on Purchasing Power Parities – Recent Advances in Methods and Applications, Washington, D.C., 30th January-2nd February, 2001.

Hill, Robert J. (2000), "Measuring Substitution Bias in International Comparisons Based on Additive Purchasing Power Parity Methods", *European Economic Review* 44: 145-162.

Hill, Robert J. (2004), "Constructing Price Indexes across Space and Time: The Case of the European Union", *American Economic Review* 94(5): 1379-1410.

Hill, Robert J. (2006a), "Superlative Index Numbers: Not All of Them Are Super", *Journal of Econometrics* 130: 25-43.

Hill, Robert J. (2006b), "When Does Chaining reduce the Paasche-Laspeyres Spread? An Application to Scanner Data", *Review of Income and Wealth* 52: 309-325.

Houthakker, H. (1950). Revealed Preference and Utility Function. *Economica*, NS 17, 159-74.

Hulten, Charles R. (1973), “Divisia Index Numbers”, *Econometrica* 41: 1017-1025.

Hulten, Charles R. (2004), “The Political Economy of Price Measurement: The NAS Report ‘At What Price’ and Beyond”, Paper prepared for the CRIW conference *Price Index Concepts and Measurement*, Vancouver, Canada, June 28-29, 2004.

Kadiyala, K. Rao (1972), “Production Functions and the Elasticity of Substitution”, *Southern Economic Journal* 38: 281-284.

Keren, Michael and J. Weinblatt (1984), “A Geometric Approach to Real Income Comparisons: A Comment on Samuelson’s ‘Analytical Notes on International Real-Income Measures’”, *Economic Journal* 94: 258-264.

Keynes, John M. (1930), *Treatise on Money, Vol. 1, The Pure Theory of Money*. New York: Harcourt Brace.

Knoblauch, Vicki (1992), “A Tight Upper Bound on the Money Metric Utility Function”, *American Economic Review* 82: 660-663.

Konüs, A.A. (1924), “Problema istinovo indeksa stoimosti zhizni”, *Ekonomicheskii Byulleten Konyunkturnovo Instituta* 3: 64-71. English translation: “The Problem of the True Index of the Cost of Living”, *Econometrica* 7 (Jan. 1939): 10-29.

Konüs, A.A. and S.S. Byushgens (1926), "K probleme pokupatelnoi cili deneg" (English translation of Russian title: "On the Problem of the Purchasing Power of Money"), *Voprosi Konyunkturi* II(1) (supplement to the *Economic Bulletin of the Conjuncture Institute*), pp. 151-172.

Koo, Anthony Y.C. (1963), “An Empirical Test of Revealed Preference Theory”, *Econometrica* 31: 646-664.

Jorgenson, Dale W. and Zvi Griliches (1967), “The Explanation of Productivity Change”, *Review of Economic Studies* 34: 249-283.

Jorgenson, Dale W. and Zvi Griliches (1971), “Divisia Index Numbers and Productivity Measurement”, *Review of Income and Wealth* , 38: 227-229.

Lau, Lawrence J. (1973), unpublished memorandum (cited in Diewert, 1976, p. 130).

McCarthy, M. D. (1967), “Approximation of the CES Production Function: A Comment”, *International Economic Review* 8: 190-192.

Manser, Marilyn E. and Richard McDonald (1988). An Analysis of Substitution Bias In Measuring Inflation, *Econometrica* 56, 909-30.

Milana, Carlo (1993), "Numeri indici", *Enciclopedia Italiana,* Institute of the Italian Encyclopedia founded by Giovanni Treccani, Rome, 5th Appendix (1979-1992), pp. 704-709.

Milana, Carlo (2001), "The Input-Output Structural Decomposition Analysis of 'Flexible' Production Systems," in Michael L. Lahr and Erik Dientzenbacher (eds.), *Input-Output Analysis: Frontiers and Extensions*, New York, Palgrave, pp. 349-80.

Milana, Carlo (2005), "The Theory of Exact and Superlative Index Numbers Revisited", Working paper no. 3, EU KLEMS Project (http://www.euklems.net/publications).

Milana, Carlo (2006), "A Net Profit Approach to Productivity Measurement, with an Application to Italy", prepared for the Workshop on productivity Measurement, 16-18 October, 2006, Bern, Switzerland, organized jointly by the OECD and the Swiss government.

Muellbauer, J.N.J. (1971), "The 'Pure Theory of the National Output Deflator' Revisited", University of Warwick, Department of Economics, Warwick Economic Research Paper Number 16, Coventry, United Kingdom.

Muellbauer, J.N.J. (1972), "The Theory of True Input Price Indices", University of Warwick, Department of Economics, Warwick Economic Research Paper Number 17, Coventry, United Kingdom.

Neary, J. Peter (2004), "Rationalizing the Penn World Table: True Multilateral Indices for International Comparisons of Real Income", *American Economic Review* 94: 1411-1428.

Pollak, Robert A. (1971) "The Theory of Cost-of-Living Index", mimeo, University Of Pennsylvania, June (Published in W. E. Diewert (ed.by), *Price Level Measurement*, Amsterdam, North-Holland Publ. Co., 1990, pp. 5-77.

Quiggin, John and Matthijs van Veelen (2007), "Multilateral Indices: Conflicting Approaches?", *Review of Income and Wealth* 53 (2), 372-78.

Reinsdorf, Marhall and Jack Triplett (2004), "A Review of Reviews: Ninety Years of Professional Thinking About the Consumer price Index and How to Measure it", Paper prepared for the CRIW conference *Price Index Concepts and Measurement*, Vancouver, Canada, June 28-29, 2004.

Russell, R. Robert (1983), "Comment on Diewert's 'The Theory of the Cost-of-Living

Index and the Measurement of Welfare Change", in W.E. Diewert and C. Montmarquette (eds.), *Price Level Measurement: Proceedings from a Conference Sponsored by Statistics Canada*, Ottawa, Minister of Supply and Services Canada, pp. 234-239.

Russell, R. R. and M. O. Tengesdal (2001). On the Power of Non-Parametric Tests of the Representative Consumer Hypothesis. Mimeo, University of California, Berkely.

Russell, R. R., Robert V. Breunig and Chia-Hui Chiu (1998). Aggregation and Econometric Analysis of Demand and Supply. In Aman Ullah and David E. A. Giles (eds.), *Handbook of Applied Economic Statistics*, New York: Marcel Dekker, Inc., 177-236.

Samuelson, Paul A. (1938a), "A Note on the Pure Theory of Consumer's Behaviour", *Economica* 5: 61-71.

Samuelson, Paul A. (1938b), "A Note on the Pure Theory of Consumer's Behaviour: An Addendum", *Economica* 5: 353-4.

Samuelson, Paul A. (1938c), "The Empirical Implications of Utility Analysis", *Econometrica* 6: 344-356.

Samuelson, Paul A. (1947), *Foundations of Economic Analysis*, Cambridge, Mass.: Harvard University Press.

Samuelson, Paul A. (1948), "Consumption Theory in Terms of Revealed Preference", *Economica* 15: 243-253.

Samuelson, Paul A. (1974), "Analytical Notes on International Real-Income Measures", *Economic Journal* 84(335): 595-608.

Samuelson, Paul A. (1984), "Second Thoughts on Analytical Income Comparisons". *Economic Journal* 94: 267-278.

Samuelson, Paul A. and Subramanian Swamy (1974), "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis", *American Economic Review* 64(4): 566-593.

Swamy, Subramanian (1984), "Samuelson's Analytical Notes on International Real-Income Measures: A Comment", *Economic Journal* 94: 265-266.

Uebe, G. (1978), "On a Flexibility Theorem of Diewert", in W. Eichhorn, R. Henn, O. Opitz, R.W. Shephard (eds), *Theory and Applications of Economic Indices*, Würzburg, Physica-Verlag, pp. 719-729.

Usher, Dan (1972), “Comment on ‘Comparative National Income of the USSR and the United States’ by Abram Bergson”, In D. J. Daly (ed. by), *International Comparisons of Price and Output*, Studies in Income & Wealth, No.37, National Bureau of Economic Research, New York: Columbia University Press, pp. 208-216.

Van Veelen, Matthijs (2002), “An Impossibility Theorem Concerning Multilateral International Comparison of Volumes”, *Econometrica* 70: 369-375.

Varian, Hal R. (1982), “The Nonparametric Approach to Demand Analysis”, *Econometrica* 50: 945-973.

Varian, Hal R. (1983), “Nonparametric Test of Models of Consumer Behavior”, *Review of Economic Studies* 50: 99-110.

Varian, Hal R. (1984), “The Nonparametric Approach to Production Analysis”. *Econometrica* 52: 579-97.

Varian, Hal R. (1985), “Non-Parametric Analysis of Optimizing Behavior with Measurement Error”, *Journal of Econometrics* 30: 445-58.

Varian, Hal R. (2006), “Revealed Preference”, in Michael Szenberg, Lall Ramrattan and Aron A. Gottesman (eds), *Samuelsonian Economics and the Twenty-First Century*. Oxford University Press.

Walsh, C.M. (1901), *The Measurement og General Exchange Value*, New York, Macmillan.

EU KLEMS WORKING
PAPER SERIES