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The Theory of Exact and Superlative Index Numbers Revisited

Working paper nr. 3

Carlo Milana

EU KLEMS WORKING PAPER SERIES

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EU KLEMS Project
Productivity in the European Union: A Comparative Industry Approach

March 2005

This project is funded by the European Commission, Research Directorate General as part of the 6th Framework Programme, Priority 8, "Policy Support and Anticipating Scientific and Technological Needs".

The Theory of Exact and Superlative Index Numbers Revisited

By Carlo Milana[0]

March 2005

Abstract

This paper proposes to clarify some important questions that are still open in the field of index number theory. The main results are the following: (*i*) the so-called Quadratic Identity on which the superlative index numbers are based can be applied in more general cases than those traditionally considered; (*ii*) it is not only the Törnqvist index number that does not rely on separability restrictions, but also some other indicators of absolute or relative changes are not based on such restrictions; (*iii*) in practice, however, all the index numbers or indicators that are considered to be superlative in Diewert's (1976) sense generally fail by construction to be really "superlative"; (*iv*) these hybrid index numbers may be far from providing the expected second-order approximation to the true index and may be found beyond the Laspeyres-Paasche interval even in the homothetic case. In conclusion, it would be more appropriate to construct a range of alternative index numbers (including even those that are not "superlative") rather than follow the common practice of searching for only one "optimal" formula.

J.E.L. Classification C43, E31, O47
KEYWORDS: Index Numbers, Separability, Aggregation

[0]*Istituto di Studi e Analisi Economica*, Piazza dell'Indipendenza, no. 4, I-00185 Rome, Italy. Phone (office): +39-06-4448-2300; Mobile: +39-347-8000984; FAX: +39-06-4448-2249; E-mail: *c.milana@isae.it.*

This paper has been prepared for the Specific Targeted Research Project "EUKLEMS-2003. Productivity in the European Union: A Comparative Industry Approach" supported by the European Commission, Research Directorate-General, within the Sixth Framework Programme, Contract No. 502049 (SCS8). Comments received from Pirkko Aulin-Ahmavaara, Alberto Heimler, and Dale W. Jorgenson on a previous version of this paper are thankfully acknowledged. The usual disclaimers apply and the views of the author do not necessarily reflect those of the mentioned persons and institutions.

"This is, I think, a case in which the mathematics itself is important, independently of the particular economic problem to which it is explicitly related".

William M. (Terence) Gorman (1959b)

1. *Introduction*

This paper revisits the theory of "exact" and "superlative" index numbers that was developed in a seminal article by Diewert (1976) and is still dominating the field of economic index numbers. An index number is said to be "exact" for a function if it is identically equal to the ratio of numerical values of that function at any pairs of points taken into comparison. The first mention of the term "exact" in the context of index numbers appeared in the *Foundations* of Paul A. Samuelson (1947, p. 155). In the introduction to the later enlarged edition of this work, Samuelson (1983, p. xx) concisely described the meaning of his contribution: "Index number theory is shown to be merely an aspect of the theory of revealed preference. [...] this is the point of revealed preference—knowledge of but two (P, Q) situations (or of a limited number of situations) can at best put bounds on each one of our sought-for ratios[1]."

In the same paragraph, he also wrote: "Thirty-five years after that analysis appeared there has been but one major advance in index number theory—namely W. E. Diewert's formalizing concept of a 'superlative index number,' which is a formula based upon two periods (p_j, q_j) data that will be exactly correct as an ordinal indicator of utility for some specified family of indifference contours. (Only a few different 'superlative' formulas are known; perhaps the set of simple superlative formulas is a limited set.)" Index numbers were, in fact, defined to be "superlative" by Diewert (1976, p. 117) (who cited Fisher, 1922, p. 247 for the use of an undefined notion of this term), if they are exact for a function that provides a second-order differential approximation to the unknown true function.

The present paper proposes to clarify the meaning of the purely mathematical *Quadratic Identity* on which the superlative numbers are based. Moreover, it generalizes and extends the cases of parameter changes where it is possible to apply the Quadratic Identity by relaxing, in particular, some

[1] In subsequent papers, Samuelson (1950, p. 24) and Samuelson and Swamy (1974, p. 585) had stressed that the underlying functions and their changes must be *homothetic* as a necessary condition for the definition of two-sided bounds for the unknown true index number.

unnecessary restrictions. It is also aimed at reformulating other basic propositions of the theory and finding solutions to some of the problems that still remain open.

Using a very general transformed quadratic function (the quadratic Box-Cox function) that encompasses well-known special cases, a unified treatment of the decomposition of functional value changes is given for the superlative index numbers. In this framework, it is shown that the Törnqvist-type index number, which is expressed in terms of relative log-changes, is not the only index number that does not rely on restrictive separability and homogeneity conditions. The Fisher "ideal" and the implicit Walsh index numbers, which are constructed (in terms of ratios) under separability and homogeneity restrictions, have counterpart indicators of relative or absolute changes that are compatible also with cases where these restrictions do not hold.

It is shown, however, that the superlative index numbers that are traditionally constructed using the observed data are not really superlative because the weights relative to at least one point of observation are not derivable from the approximating quadratic function for which they are intended to be exact. As a consequence, the traditional formulae used to construct superlative index numbers are, in fact, hybrid index numbers that may be far from providing the expected second-order approximation.

The results obtained have various consequences in the theory and practice of index numbers. Firstly, they help us to explain the empirical results recently obtained by Hill (2002, 2005) (and mentioned by Diewert, 2004, pp. 450-451) concerning the large spread in numerical values of alternative superlative index numbers. In another previous seminal paper, Diewert (1978) had shown that they are expected to approximate each other up to the second-order and to be numerically very close if the two points under comparison do not vary very much[2]. However, it has been surprising to find empirically that the spread between the largest and the smallest superlative index numbers often exceeds that between the Laspeyres and Paasche indexes, which are usually considered to be the bounds of the interval of possible values of economic index numbers, at least in homothetic cases. In the Hill's (2002, 2005) empirical applications, the largest and the smallest superlative index numbers have resulted to differ by more than 100 per cent for a standard US national data set and by about 300 per cent in a cross-

[2]See also Vartia (1978). In a subsequent paper, Allen and Diewert (1981) indicated theoretical and numerical bounds for these index numbers and stated that the choice of the index number formula will not matter much if these bounds are narrow. However, Diewert (1978, p. 890, fn. 8), citing the discussion in Lau (1974, p. 183), had already recognized that, without performing extensive computations involving the third order partial derivatives of the index number formulae, we cannot specify exactly how small should be the change between the two compared points in order for the superlative indexes to be all very close to each other.

section comparison of countries based on an OECD data set[3]. The present paper provides an additional explanation of this result.

It has been shown that the only index number that always falls numerically between the Laspeyres and Paasche indexes is the Fisher "ideal" index number. This always falls between the former two index numbers just because it consists of their geometric average. Summarizing our results, it turns out that: (*i*) the so-called Quadratic Identity on which the superlative index numbers are based can be applied in more general cases than those traditionally considered; (*ii*) it is not only the Törnqvist index number that does not rely on separability restrictions, but also some other indicators of absolute or relative changes are not based on such restrictions; (*iii*) in practice, however, all the index numbers or indicators that are considered to be superlative in Diewert's (1976) sense generally fail by construction to be really "superlative"; (*iv*) these hybrid index numbers may be far from providing a second-order approximation to the true index number and may be found beyond the Laspeyres-Paasche interval even in the homothetic case. Since the degree of approximation of the available index number formulae cannot be assessed, all of these are equally valid candidates as a good approximation to the "true" unknown index number.

The paper is organized as follows. Section 2 re-examines the Quadratic Identity within a general framework of accounting for functional value changes of an arbitrary differentiable function. Section 3 extends this analytical approach to the case where the parameters or even the functional forms of two functions under comparison may differ and generalizes the results obtained in the literature up till now. Section 4 extends the analysis further by using transformed functions. Section 5 provides a unified approach to index numbers by using a general transformed quadratic function with no *a priori* separability and homogeneity restrictions. Section 6 deals with the approximation properties of index numbers and establishes conditions for constructing truly superlative index numbers. Section 7 concludes with remarks and suggestions on the use of index numbers.

2. *General formulation of the quadratic approximation lemma*

In the general case of an arbitrary differentiable function, the following result is obtained:

[3] Allen and Diewert (1981, p. 430) had clearly recognized that "in many applications involving the use of cross section data or decennial census data, there can be a tremendous amount of variation in prices or in quantities between the two periods so that alternative superlative index number can generate quite different results".

LEMMA 2.1. Accounting for Functional Value Differences. *Let z be a vector of N real valued variables and let us assume that an arbitrary function $f(z)$ is continuously differentiable at least once, then, for all z^0 and z^1,*

$$f(z^1) - f(z^0) \quad = \quad \left[(1-\theta)\nabla_z f(z^0) + \theta \nabla_z f(z^1)\right]^T (z^1 - z^0) \qquad (2.1)$$

where $\nabla_z f(z^t)$ is the gradient vector of f evaluated at z^t; and, denoting with $R_1^0(z^0, z^1)$ and $R_1^1(z^0, z^1)$ the remainder terms associated with the polynomials of order one in the Taylor series expansion for f around z^0 and z^1, respectively, θ takes the particular value $\theta^*(z^0, z^1) \equiv \frac{R_1^0(z^0,z^1)}{R_1^0(z^0,z^1)+R_1^1(z^0,z^1)}$ when $R_1^0(z^0, z^1) + R_1^1(z^0, z^1) \neq 0$, or θ takes any real number as a value if f is linear in z ($R_1^0(z^0, z^1) = R_1^1(z^0, z^1)$).

Proofs of propositions are given in Appendix B.

It is straightforward to show that the weight θ in (2.1) falls within the interval $0 \leq \theta \leq 1$ if $f\ (z)$ is quasiconcave or quasiconvex. Lemma (2.1) can be complemented with the following corollaries.

COROLLARY 2.1. Accounting for Functional Value Ratios. *If an arbitrary function $f(z)$ is homothetic, so that $f(z) = F[\phi(z)]$, where $F(\cdot)$ is a well behaved transformation function (real valued, continuously differentiable, monotonically increasing and quasiconcave) and $\phi(z)$ is also well behaved and linearly homogeneous*[4]*, then, for all z^0 and z^1,*

$$\frac{f(z^1)}{f(z^0)} \quad = \quad I_Z \cdot I_Y \qquad (2.2)$$

where

$$I_Z \quad \equiv \quad \frac{\theta + (1-\theta)\sum_{i=1}^N s_i^0 \frac{z_i^1}{z_i^0}}{(1-\theta) + \theta \sum_{i=1}^N s_i^1 \frac{z_i^0}{z_i^1}} \; = \frac{\phi(z^1)}{\phi(z^0)} \qquad (2.3)$$

$$I_Y \quad \equiv \quad \frac{f(z^1)}{f(z^0)} \Big/ \frac{\phi(z^1)}{\phi(z^0)} \qquad (2.4)$$

[4] The concept of homotheticity was explicitly spelled out by Shephard (1953) and Malmquist (1953), although earlier researchers as Frisch (1936, p. 25) and Samuelson (1950, p. 24) had dealt with it implicitly.

with θ *being defined by Lemma (2.1) and* $s_i^t \equiv \frac{\partial f(z^t)}{\partial z_i^t} \cdot z_i^t / \sum_{i=1}^N \frac{\partial f(z^t)}{\partial z_i^t} \cdot z_i^t$ $(t = 0, 1)$.

I_Z and I_Y represent, respectively, the aggregator index of z and the index of scale effects. The index I_Z is *exact* for the aggregator or index function ϕ, which means that it is identically equal to the ratio $\phi(z^1)/\phi(z^0)$[5]. It is homogeneous of degree one in z and is equal to a Laspeyres-type index number if $\theta = 0$, whereas it is equal to a Paasche-type index number if $\theta = 1$. It is straightforward to show that, if also the function ϕ is well behaved (in particular, quasiconvex or quasiconcave), then the Laspeyres- and Paasche-type index numbers mentioned above are the bounds of the interval of all possible numerical values taken by the ratio $\phi(z^1)/\phi(z^0)$ for all z^0 and z^1.

Let us define the quadratic function:

$$\begin{aligned} f_Q(z) &\equiv a_0 + a^T z + \frac{1}{2} z^T A\, z \\ &= a_0 + \sum_{i=1}^N a_i z_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} z_i z_j, \end{aligned} \tag{2.5}$$

where the a_i, a_{ij} are constant parameters and $a_{ij} = a_{ji}$ for all i, j. .

The following well-known result can be derived as another corollary of Lemma 2.1:

COROLLARY 2.2. Diewert's (1976, p. 117) Quadratic Identity. *If and only if* $f(z)$ *has the functional form of the quadratic function* $f_Q(z)$ *defined by (2.5) where* $A \neq 0_{N \times N}$, *then the weight* θ *in (2.1) is equal to 1/2 for all* z^0 *and* z^1, *so that*

$$f_Q(z^1) - f_Q(z^0) = \frac{1}{2}[\nabla_z f_Q(z^0) + \nabla_z f_Q(z^1)]^T (z^1 - z^0) \tag{2.6}$$

Some observations concerning this result are in order:

(i) The inequality $\nabla_z f_Q(z^0) \neq \nabla_z f_Q(z^1)$ is sufficient to infer that the function is non-linear in z (the converse is, however, not true since a non-linear function may have equal first derivatives at some different points).

(ii) Diewert (1976, p. 117) formulated the Quadratic Identity, which he called "Quadratic approximation lemma", under the assumption that

[5] The terms "index function" and "aggregator function" can be used here interchangeably. The former was used by Shephard (1953, pp. 47-49) and Solow (1956, pp. 102-106), whereas the latter was used by Diewert (1976).

the examined function is thrice continuously differentiable. This implies, In particular, that f is not linear in z.

(iii) Lau (1979), noted that "generalizing Diewert's proof to the twice continuously differentiable case is straightforward" (p. 74, fn. 1). He also gave an alternative proof of the Quadratic Identity by assuming a once differentiable function and observed that this "widens considerably the applicability of the lemma and consequently of the results which depend on its validity" (p. 74).

(iv) Under the general once differentiability assumption and no additional non-linearity condition, the *quadratic* function (2.5) can in fact be regarded as having the most general functional form with which equation (2.6) exactly holds. This equation still holds in special (limit) cases including the *linear* function $f_L(z) \equiv a_0 + a^T z$, corresponding to f_Q with $A = 0_{N\times N}$, and the *constant-value* $f_C(z) \equiv a_0$, corresponding to an f_Q with $a = 0_N$ and $A = 0_{N\times N}$.

(v) The necessity part of Corollary (2.2) means that, when a function is indeed non-linear (as implied by the thrice differentiability condition), then equation (2.6) is compatible only with a "strictly" quadratic functional form, corresponding to an f_Q with $A \neq 0_{N\times N}$ (in this case, f_Q can be said to belong to the class of "strongly" concave or convex functions, using Avriel's *et. al.*, 1988, pp. 1-2, terminology).

The rationale of the Quadratic Identity established by Corollary (2.2) can be explained geometrically as in Figure 1, where a quadratic function in one single variable is represented. Let the functions ${f_L}^0(z)$ and ${f_L}^1(z)$ be, respectively, the first-order polynomials of the Taylor series expansions for $f_Q(z)$ around z^0 and z^1, that is

$$\begin{aligned} {f_L}^t(z) &\equiv a^t + b^t z \\ &= f_Q(z^t) + f_Q'(z^t)(z - z^t) \qquad \text{for } t = 0, 1 \end{aligned} \tag{2.7}$$

where prime means differentiation, $a^t \equiv f_Q(z^t) - f_Q'(z^t)\, z^t$, and $b^t \equiv f_Q'\,(z^t)$.

Taking the first differences yields

$${f_L}^t(z) - {f_L}^t(z^t) \;=\; f_Q'(z^t)(z - z^t) \qquad \text{for } t = 0, 1 \tag{2.8}$$

In Figure 1, ${f_L}^0(z^1) - {f_L}^0(z^0) = AB'$ and ${f_L}^1(z^1) - {f_L}^1(z^0) = A'B$.

The first differences of $f_Q(z)$ can be derived in terms of the Taylor series expansion for f around z^0 or z^1 as follows:

$$\begin{aligned}
f_Q(z^1) - f_Q(z^0) &= f_Q{}'(z^0)(z^1 - z^0) + \frac{1}{2} f_Q{}''(z^0)(z^1 - z^0)^2 \\
&= f_L^0(z^1) - f_L^0(z^0) + \frac{1}{2} f_Q{}''(z^0)(z^1 - z^0)^2 \qquad (2.9) \\
&\quad \text{using (2.8)}
\end{aligned}$$

$$\begin{aligned}
f_Q(z^0) - f_Q(z^1) &= f_Q'(z^1)(z^0 - z^1) + \frac{1}{2} f_Q{}''(z^1)(z^0 - z^1)^2 \\
&= f_L^1(z^0) - f_L^1(z^1) + \frac{1}{2} f_Q{}''(z^0)(z^1 - z^0)^2 \qquad (2.10) \\
&\quad \text{using (2.8)}
\end{aligned}$$

Multiplying through equation (2.10) by -1 and rearranging terms yield

$$\begin{aligned}
f_Q(z^1) - f_Q(z^0) &= f_Q'(z^1)(z^1 - z^0) - \frac{1}{2} f_Q''(z^1)(z^0 - z^1)^2 \\
&= f_L^1(z^1) - f_L^1(z^0) - \frac{1}{2} f_Q''(z^0)(z^1 - z^0)^2 \qquad (2.11)
\end{aligned}$$

The second derivative $f_Q''(z)$ is a constant and has a negative (positive) algebraic sign if f_Q is concave (convex). Hence,

$$\begin{aligned}
&\frac{1}{2} f_Q''(z^0)(z^1 - z^0)^2 \qquad (BB' \quad \text{in Figure 1}) \\
= \; &\frac{1}{2} f_Q''(z^1)(z^0 - z^1)^2 \qquad (AA' \quad \text{in Figure 1}) \qquad (2.12)
\end{aligned}$$

In view of (2.12), the arithmetic average of (2.9) and (2.11) leads us to

$$\begin{aligned}
f_Q(z^1) - f_Q(z^0) &= \frac{1}{2}[f_Q'(z^1) + f_Q'(z^0)]\ (z^1 - z^0) \\
&= \frac{1}{2}\left\{\left[f_L^0(z^1) - f_L^0(z^0)\right] - \left[f_L^1(z^1) - f_L^1(z^0)\right]\right\} \text{ using (2.8)} \\
&= \frac{1}{2}\left\{\left[AB'\right] + \left[A'B\right]\right\} \quad \text{in Figure 1} \\
&= \frac{1}{2}\left\{\left[AB + AA'\right] + \left[AB - BB'\right]\right\} \\
&\qquad (BB' < 0 \text{ and } AA' < 0 \text{ since } f_Q \text{ is concave}) \\
&= AB \quad \text{in view of (2.12)} \qquad (2.13)
\end{aligned}$$

Figure 1. Geometrical representation of Diewert's (1976) Quadratic Identity

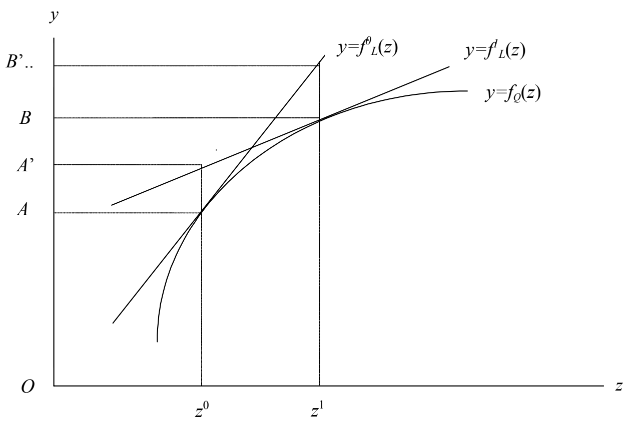

Figure 1:

Since the function is quadratic, the remainder terms AA' and BB' of first-order approximations are equal and therefore they completely offset each other. In the general case of a non-quadratic function f, these remainder terms may differ substantially and $f(z^1) - f(z^0) = AB + (AA' - BB')$ may turn out to be very different from AB.

The following useful result is obtained in terms of ratios rather than differences by imposing more restrictive conditions on the quadratic function:

COROLLARY 2.3. Accounting for Functional Value Ratios of a Quadratic Homothetic Function (Byushgens, 1925; Konüs and Byushgens, 1926; Frisch, 1936, p. 30; Wald, 1939, p. 331; Pollak, 1971; Afriat, 1972, p. 45; 1977, pp. 141-143; 2005, pp. 177-178). *If a continuously once-differentiable quadratic function $f_Q(z)$ defined by (2.5) is a homothetic transformation of a linearly homogeneous function, so that $a_i = 0$ for $i = 0, 1, ..., N$ and $f_Q(z) = [\phi(z)]^2$, where $\phi(z) \equiv [\frac{1}{2}z^T Az]^{1/2}$, then, for all z^0 and z^1,*

$$\frac{f_Q(z^1)}{f_Q(z^0)} = I_Z \cdot I_Y \tag{2.14}$$

where

$$I_Y \quad = \quad I_Z \equiv \left[\frac{\sum_{i=1}^{N} s_{Q,i}^{0} \frac{z_i^1}{z_i^0}}{\sum_{i=1}^{N} s_{Q,i}^{1} \frac{z_i^0}{z_i^1}} \right]^{1/2} \tag{2.15}$$

with $s_{Q,i}^t \equiv \frac{\partial f_Q(z^t)}{\partial z_i^t} \cdot z_i^t / \sum_{i=1}^N \frac{\partial f_Q(z^t)}{\partial z_i^t} \cdot z_i^t$ *(*$t = 0, 1$*).*

We note that I_Y and I_Z represent, respectively, the aggregator index of z and the index of scale effects. they correspond to an ideal Fisher-type index number, which is "exact" (identically equal to) the ratio $\phi(z^1)/\phi(z^0) = [\frac{1}{2}z^{1T}Az^1]^{1/2}/[\frac{1}{2}z^{0T}Az^0]^{1/2}$ and is linearly homogeneous in z.

The accounting framework of the Quadratic Identity is usually seen as particularly convenient for a practical approximation of the change in $f(z)$ when this function is unknown but its first derivatives with respect to z are somehow observable or measurable. The Bernstein-Weierstrauss approximation theorem states that, on a closed bounded domain, a continuous function can be uniformly approximated by polynomials. The function $f_Q(z)$ defined by (2.5) can be seen as providing a second-order approximation to an arbitrary true function $f(z)$ around the point z^* when its parameters are "calibrated" to certain numerical values, so that $f_Q(z^*) = f(z^*)$, $\nabla f_Q(z^*) = \nabla f(z^*)$, and $\nabla^2 f_Q(z^*) = \nabla^2 f(z^*)$[6][7]. However, the error of approximation that is obtained by applying formula (2.6) using the "observed" weights $\nabla_z f(z^0)$ and $\nabla_z f(z^1)$ is different from the error of second-order approximation obtainable using the weights $\nabla_z f_Q(z^0)$ and $\nabla_z f_Q(z^1)$. This can be assessed more clearly by means of the following result:

LEMMA 2.2. General Quadratic Approximation Lemma. *Let $f(z)$ be an arbitrary once differentiable function. If the value change of the arbitrary*

[6] Lau (1974, pp. 183-184) distinguished the concept of "second-order differential approximation" from that of "second-order numerical approximation" and claimed that polinomial expressions like (2.5), which can be interpreted as Taylor's series expansions up to the second order, provide both types of approximation (see also Barnett, 1983, pp. 19-20 for further discussion).

[7] A "problem of accuracy of approximation" arises with expressions like (2.5). Fuss, McFadden, and Mundlak (1978, pp. 233-234) clearly stated this problem in the following terms: "If a flexible form is calibrated to provide a second-order approximation at a point, then the approximation is of this order only in a small neighborhood of this point. In other regions of interest, the form may be a poor approximation to the true function. [...] Further, the qualitative implications of the calibrated approximation may depend on the point of approximation; this is true, for example, of separability, which involves properties of the true function beyond second-order".

$f(z)$ is accounted for by using (2.1) where θ is set equal to 1/2 as in (2.6), then the obtained approximation error is equal to the difference of two first-order approximations multiplied by $(0.5-\theta)$*, that is*

$$f(z^1)-f(z^0) = \frac{1}{2}[\nabla_z f(z^0)+\nabla_z f(z^1)]^T(z^1-z^0) + \textit{Error of approximation} \quad (2.16)$$

where

$$\textit{Error of approximation} \equiv \left(\frac{1}{2}-\theta\right)\left\{\left[f_L^0(z^1)-f_L^0(z^0)\right]-\left[f_L^1(z^1)-f_L^1(z^0)\right]\right\} \quad (2.17)$$

and, for $r=0,1$, $f_L^r(z)$ is a first-order approximating (linear) function that is tangent to $f(z)$ at z^r, that is $f_L^r(z) \equiv a^r + b^r z = f(z^r) + \nabla_z f\ (z^r)^T(z-z^r)$, with $a^r = f(x^r) - \nabla_z f\ (x^r)\ x^r$ and $b^r = \nabla_z f\ (x^r)$.

The error of approximation represented by (2.17) is made of the difference between two *linear* approximations to $f(z^1)-f(z^0)$, which are constructed, respectively, around z^0 and z^1. If f is linear in z, then $f_L^0(z) = f_L^1(z)$, whereas the term in brace brackets is equal to 0; if f is quadratic, then $\frac{1}{2}-\theta=0$; if f has any other functional form, then, in general, $\theta \neq \frac{1}{2}$ and the error of approximation may turn out to be non-negligible. Its size depends on the functional form of f and the distance of z^1 from z^0. This is shown in the following numerical example.

EXAMPLE 2.1. Let $f(z)$ be the following cubic function of one single variable:

$$f(z) \equiv a + b\ z + \frac{1}{2}c\ z^2 + \frac{1}{6}d\ z^3 \quad (2.18)$$

where $a=15.5008794$, $b=-38.2764683$, $c=65.4734033$, $d=-53.7666768$ so that $f(z)=1.0$ with $z^0=1.0$ and $f(z)=1.5$ with $z^1=1.5$. The first derivative is given by

$$f^{\,'}(z) = b + c\ z + \frac{1}{2}d\ z^2 \quad (2.19)$$

A second order differential approximation to $f(z)$ around $z^0=1.0$ is given by the following quadratic function

$$f_Q(z) \equiv \alpha + \beta\ z + \frac{1}{2}\gamma\ z^2 \quad (2.20)$$

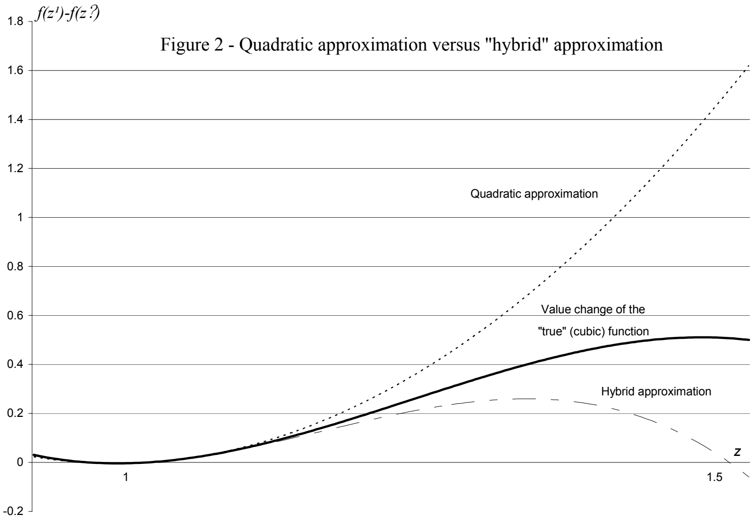

Figure 2:

where $\alpha = 6.5397667$, $\beta = -11.3931299$, and $\gamma = 11.7067265$ so that

$$f(z^0) = f_Q(z^0) \tag{2.21}$$

$$f'(z^0) = f_Q^{'}(z^0) \tag{2.22}$$

$$f''(z^0) = f_Q^{''}(z^0) \tag{2.23}$$

In general, these equalities do not hold at $z^1 \neq z^0$. Table 1 compares, for different changes in z, the actual value changes in f which can be computed exactly using formula (2.1), that is $\frac{1}{2}[(1-\theta)f'(z^0) + \theta f'(z^1)]^T (z^1 - z^0)$ (see column 6) with their "hybrid" approximation obtained by imposing $\theta = 1/2$ in (2.1), that is $\frac{1}{2}[f'(z^0) + f'(z^1)]^T (z^1 - z^0)$ (see column 7). Figure 2 compares this "hybrid" approximation and the quadratic approximation given by identity (2.6), that is $\frac{1}{2}[f_Q'(z^0) + f_Q'(z^1)]^T (z^1 - z^0)$ for different values of z^1.

We note, at the end of this section, that an alternative procedure can be set up using a Divisia index in order to account for value changes in the functions taken into exam here. Under the hypotheses defined above, in principle, this procedure should lead us to the same results that we have obtained thus far[8].

[8] See Milana (1993) for alternative index numbers implementing the Divisia index in the discrete.

Table 1 – Functional value change computed using (2.1) with alternative weights

z^0	z^1	$f'(z^0)$	$f'(z^1)$	$\theta = \theta^*$	$f(z^1) - f(z^0)$ (actual value, computable using (2.1) with $\theta = \theta^*$)	$f(z^1) - f(z^0)$ (approximated using (2.1) with θ = 1/2)	Error of approximation
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8) = (7)-(6)
1.0	1.125	0.313597	1.356885	0.567104	0.113156	0.044051	0.069105
1.0	1.250	0.313597	1.560070	0.724664	0.304217	0.235982	0.068235
1.0	1.375	0.313597	0.923150	1.533673	0.468169	0.231890	0.236279
1.0	1.5	0.313597	-0.553875	-0.791270	0.5	-0.060070	0.560070

Figure 3:

3. Accounting for differences in functional values when parameters or functional forms also differ

Let us consider two functions differing in functional forms or in parameters, some of which being functions of other variables, that is, for $t = 0, 1$,

$$f^{\,t}(z) \quad \equiv \quad h^t(z, k^t) \tag{3.1}$$

where some parameters of f^t are functions of an M-dimensional vector k and others may differ autonomously. Special cases of (3.1) are of some interest for the following discussion. If h^t is separable in z, then, in the case of "weak" ("non-linear") separability, (3.1) can be rewritten as:

$$h^t(z, k^t) \quad = \quad h^t_s\left[\zeta(z), k^t\right] \tag{3.2}$$

If the function ζ is homothetic (homogeneous), then h^t is said to be homothetically (homogenously) weakly separable. If all functions in (3.2) are homothetic (homogeneous), then h^t can be expressed as:

$$h^t(z, k) \quad = \quad \kappa^t(k) \cdot \zeta(z) \tag{3.3}$$

which is equivalent to

$$h^t(z, k) \quad = \quad H^t \cdot [\kappa^{*t}(k) + \zeta^*(z)], \tag{3.4}$$

a strongly (additively) separable function, where the starred functions differ in parameter values from the respective counterparts in (3.3). More special cases can be defined where the function (3.1) is separable also in k, so that $\kappa^t(k) \equiv \sigma^t \cdot \kappa(k)$[9].

We recall that separability is a necessary but not a sufficient condition for constructing aggregates. An aggregate of separable variables exists if the index function of those variables is also homogeneous of degree one. In our case, we should have $\lambda\zeta(z) = \zeta(\lambda z)$ for all z and $\lambda\kappa(k) = \kappa(\lambda k)$ for all k in order for ζ and κ to be aggregator functions. In the producer context, Shephard (1953, p. 63) was the first to realize the necessity of imposing the linear homogeneity property on the index functions of separable quantity variables in order for the respective dual price index functions to be independent of other prices and quantities[10].

The concept of *homothetic separability* has been introduced by Shephard (1953, p. 43) and Blackorby, Lady, Nissen, and Russell (1970) in the producer and the consumer contexts, respectively. They assumed homothetic functions within weakly separable functions. This definition is related to the concept of *homogeneous separability*, which is obtained under linear homogeneity restrictions on separable functions, regarded also as aggregability conditions (see, for example, Green, 1964, p. 25)[11].

Let us now consider the following results:

LEMMA 3.1. Accounting for Functional Value Differences when Parameters or Functional Forms Differ. *If* $f^0(z) \equiv h^0(z, k^0)$ *and* $f^1(z) \equiv h^1(z, k^1)$ *are two arbitrary functions differentiable at least once and characterized by*

---

[9] The concept of separability was independently proposed by Sono (1945) and Leontief (1947a, 1947b) in the consumer and the producer contexts, respectively. The terminology of "weak" and "strong" separability was introduced by Strotz (1959). More specifically, a function $F(x) = F(x^1, x^2, ..., x^M)$ is said to be "weakly" separable in the partition $(x^1, x^2, ..., x^M)$ if there exist functions $F^*, F^1,...,F^M$ such that $F(x) = F^*[F^1(x^1), F^2(x^2), ...F^M(x^M)]$, and it said to be "strongly" separable if there exist functions $F^{**}, F^1,...,F^M$ such that $F(x) = F^{**}[F^1(x^1) + F^2(x^2) + ... + F^M(x^M)]$.

[10] Solow (1956, p. 104n) attributed to Samuelson the proof that, in a two stage maximization procedure, the aggregating quantity index functions must be linearly homogenous. However, Gorman (1959a, pp. 476-478) found, in some special cases, conditions weaker than linear homogeneity for a function of separable variables to be an aggregator function (see also Blackorby, Primont, and Russell, 1978, Ch. 5, Blackorby, Schworm, and Fisher, 1986, and Blackorby and Schworm, 1988).

[11] More specifically, this condition is referred to as *homogeneous weak separability* if the function is weakly separable as in (3.2) (see, for example, Diewert, 1993, pp. 12-13 and p. 28, and Diewert and Wales, 1995, pp. 260-261). Similarly, we may use the term *homogeneous strong separability* if the function is linearly (or log-linearly) separable as in (3.3)-(3.4).

different parameter values or functional forms, then,

$$f^1(z^1) - f^0(z^0) = \left[(1-\theta)\nabla_z f^0(z^0) + \theta\nabla_z f^1(z^1)\right](z^1 - z^0) \quad (3.5)$$
$$+ \text{"technical" change component (TC)}$$

where

$$TC \equiv \theta\left[f^1(z^0) - f^0(z^0)\right] + (1-\theta)\left[f^1(z^1) - f^0(z^1)\right] \quad (3.6)$$

and, by defining R_1^0 and R_1^1 *as the remainder terms associated with the first-order polinomials in the Taylor series expansion of* $f^1(z^1)$ *and* $f^0(z^0)$ *around* z^1 and z^0, *respectively, so that* $\theta = \theta^* \equiv \frac{R_1^0}{R_1^0 + R_1^1}$ *when* $R_1^0 + R_1^1 \neq 0$ *or* θ *is a real number that takes any value if* $R_1^0 = R_1^1 = 0$ (if both $f^0(z)$ and $f^1(z)$ are linear in z).

It is worth noting that the weight θ in (3.5) falls within the interval $0 \leq \theta \leq 1$ if $f\ (z)$ is quasiconcave or quasiconvex.

Lemma (3.1) can be complemented with the following corollaries.

COROLLARY 3.1. Accounting for Functional Value Ratios between Arbitrary Differentiable Functions. *If two arbitrary continuously differentiable functions* $f^0(z)$ and $f^1(z)$ *are homothetically separable in* z *(so that* $f^t(z) \equiv \sigma^t F[\phi(z)]$, *for* $t = 0, 1$, *where* $F(\cdot)$ *is a well behaved transformation function and* $\phi(z)$ *is linearly homogeneous, then, for all* z^0 *and* z^1,

$$\frac{f^1(z^1)}{f^0(z^0)} = I_Z \cdot I_Y \cdot I_T \quad (3.7)$$

where

$$I_Z = \frac{\theta + (1-\theta)\sum_{i=1}^N s_i^0 \frac{z_i^1}{z_i^0}}{(1-\theta) + \theta\sum_{i=1}^N s_i^1 \frac{z_i^0}{z_i^1}}, \quad (3.8)$$

with θ *defined in Lemma (2.1), and* $s_i^t \equiv \frac{\partial f^t(z^t)}{\partial z^t} \cdot z_i^t / \sum_{i=1}^N \frac{\partial f^t(z^t)}{\partial z^t} \cdot z_i^t$, *for* $t = 0.1$,

$$I_T \equiv \frac{\sigma^1}{\sigma^0} \quad (3.9)$$

$$I_Y = \frac{f^1(z^1)}{f^0(z^0)}/(I_Z \cdot I_T) \qquad (3.10)$$

The index numbers I_Z, I_Y, and I_T represent, respectively, the aggregator index of z, the index of scale effects, and the index of changes in parameters or functional form. In particular, the index I_Z is homogeneous of degree one in z and is exact for the function ϕ, since it is identically equal to the ratio $\phi(z^1)/\phi(z^0)$. It is a Laspeyres-type index number if $\theta = 0$, whereas it is a Paasche-type index number if $\theta = 1$. It is straightforward to show that, if the function $\phi(z)$ is quasiconvex or quasiconcave, then the Laspeyres- and Paasche-type index numbers are the bounds of the interval of all possible numerical values taken by the ratio $\phi(z^1)/\phi(z^0)$ for all z^0 and z^1.

If $f^1 \equiv h^1$ and $f^0 \equiv h^0$ are defined under the homothetic separability as in (3.3), then the ratio f^1/f^0 can be decomposed into an index number of weighted ratios z_i^1/z_i^0, an index of scale effects, and an index number of weighted changes in parameters or functional form, as shown in (3.7)-(3.8). By contrast, no separability restriction, however, is required to decompose the difference $f^1 - f^0$ (or $h^1 - h^0$).

Let us define the quadratic function:

$$\begin{aligned} f_Q^t(z) &\equiv a_0^t + a^t\, z + \frac{1}{2} z\, A^t z \\ &= a_0^t + \sum_{i=1}^N a_i^t z_i + \frac{1}{2}\sum_{i=1}^N \sum_{j=1}^N a_{ij}^t z_i z_j \qquad (3.11) \end{aligned}$$

where all parameters are variable. Moreover, the parameters of f_Q^t are function themselves of other variables as follows:

$$a_0^t \equiv \alpha_0^t + k^{tT} \cdot \beta^t + \tfrac{1}{2} k^{t\,T} \cdot B^t \cdot k^t,$$
$$a^t \equiv \alpha^t + k^{tT} \cdot \Gamma^t$$

so that

$$f_Q^t(z) \equiv h_Q^t(z, k^t)$$

where

$$\begin{aligned} h_Q^t(z,k) \equiv\ & \alpha_0^t + \alpha^{tT}\, z + \frac{1}{2} z^T A^t z \\ & + k^T \beta^t + \frac{1}{2} k^T B^t k \\ & + k^T \Gamma^t z. \qquad (3.12) \end{aligned}$$

The function $h_Q^t(z^t, k)$ can be expressed also as a quadratic function in k:

$$
\begin{aligned}
h_Q^t(z^t, k) &\equiv \\
\psi_Q^t(k) &\equiv b_0^t + k^T b^t + \frac{1}{2} k^T B^t k
\end{aligned}
\tag{3.13}
$$

where $b_0^t \equiv \alpha_0^t + \alpha^{tT} \; z^t + \frac{1}{2} z^{tT} A^t z^t$, $b^t \equiv \beta^t + \Gamma^t z^t$.

The quadratic functional form (3.11)-(3.12) may be reformulated in order to represent special separability cases. The early literature on separability of the so-called "flexible" functions, which provide a second-order approximation to an arbitrary function, has indicated the required parameter restrictions[12]. The "weak" separability of f_Q^t in z with respect to parameter changes occurs if (3.11) can be rewritten as:

$$
\begin{aligned}
f_Q^t(z) &\equiv f_Q^t[\zeta(z)] \\
&\equiv a_0^t + a_1^t \zeta(z) \; + \frac{1}{2} \zeta(z) \, a_2^t \zeta(z)
\end{aligned}
\tag{3.14}
$$

where a_1^t and a_2^t are variable scalar parameters and $\zeta(z)$ must be the linear function $\alpha^T z$ in order for the function f_Q^t to have the functional form (3.11). In this case, h_Q^t can be rewritten as

$$
\begin{aligned}
h_Q^t(z, k) &\equiv h_Q^t \left[\zeta(z), k\right] \\
&= \alpha_0^t + \alpha_1^t \zeta(z) + \frac{1}{2} \zeta(z) a_2^t \zeta(z) \\
&\quad + k^T \beta^t + \frac{1}{2} k^T B^t k + \frac{1}{2} k^T \gamma^t \zeta(z)
\end{aligned}
\tag{3.15}
$$

where γ^t is an M-dimensional (column) vector of variable parameters. A similar condition can be imposed on the parameters of the first- and second-order terms in k in order to represent the case of "non-linear" separability of h_Q^t in k, by defining the linear function $\kappa(k) \equiv \beta^T k$. Therefore, by imposing the "weak" separability conditions for z and k, h_Q^t becomes a *non-linear* (quadratic) function whose arguments are the linear functions $\zeta(z)$ and $\kappa(k)$, that is $h_Q^t \left[\zeta(z), \kappa(k)\right] \equiv \alpha_0^t + \alpha_1^t \zeta(z) + \frac{1}{2} \zeta(z) a_2^t \zeta(z) + \kappa(k) \beta_1^t + \frac{1}{2} \kappa(k) \beta_2^t \kappa(k)$ $+ \frac{1}{2} \kappa(k) \gamma^t \zeta(z)$.

The "strong" or "linear" separability of $f_Q^t \equiv h_Q^t$ in z, k, and parameter

[12] See Diewert (1993, p. 15) for references to this literature.

changes occurs when $\Gamma^t = 0_{M\times N}$ in (3.12) so that

$$f_Q^t(z) \equiv h_Q^t(z, k^t) = \sigma^t[\zeta_Q(z) + \kappa_Q(k)] \tag{3.16}$$

where $\zeta_Q(z) = \alpha_0 + \alpha z + \frac{1}{2}z\,Az$, $\kappa_Q(k) = k^T\beta + \frac{1}{2}k^T Bk$, and σ^t is a factor of proportionality. We shall see that, under the conditions of homogeneity and the necessary restrictions on parameter values, the form of (3.16) is equivalent to

$$f_Q^t(z) \equiv h_Q^t(z, k^t) = \sigma^t \cdot \zeta_Q^*(z) \cdot \kappa_Q^*(k) \tag{3.17}$$

where which is indistinguishable from the "weak separability" case with only two separable groups of variables[13].

The following results regarding the accounting for functional value differences can be obtained:

COROLLARY 3.2. Accounting for Numerical Value Differences of Two Quadratic Functions Differing in Parameters. *If two quadratic functions* $f_Q^1(z^1)$ *and* $f_Q^0(z^0)$ *defined by (3.11)-(3.12) differ in parameters, then, for all* z^0 and z^1,

$$\begin{aligned} f_Q^1(z^1) - f_Q^0(z^0) &= \frac{1}{2}[\nabla_z f_Q^0(z^0) + \nabla_z f_Q^1(z^1)](z^1 - z^0) \\ &\quad + \textit{Parameter change component} \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} &\textit{Parameter change component} \\ \equiv\ &(a_0^1 - a_0^0) + (a^1 - a^0)^T\frac{1}{2}(z^1 + z^0) + \frac{1}{2}z^0(A^1 - A^0)z^1 \end{aligned} \tag{3.19}$$

or

$$\begin{aligned} h_Q^1(z^1, k^1) - h_Q^0(z^0, k^0) &= \frac{1}{2}[\nabla_z h_Q^0(z^0, k^0) + \nabla_z h_Q^1(z^1, k^1)](z^1 - z^0) \\ &\quad + \textit{Residual component} \end{aligned} \tag{3.20}$$

[13] This is a well known result (see Gorman, 1959a, p. 471).

the residual component is equal to the parameter change component defined by (3.19), in which the sum of the "zero-" and "first-order" terms can be, in turn, decomposed as follows

$$
\begin{aligned}
& (a_0^1 - a_0^0) + (a^1 - a^0)^T \frac{1}{2}(z^0 + z^1) \\
= \; & \frac{1}{2}\left[\nabla_k h^0(z^0, k^0) + \nabla_k h^1(z^1, k^1)\right](k^1 - k^0) \\
& + (\alpha_0^1 - \alpha_0^0) + (\alpha^1 - \alpha^0)\frac{1}{2}(z^0 + z^1) \\
& + \frac{1}{2}(k^0 + k^1)(\beta^1 - \beta^0) + \frac{1}{2}k^0(B^1 - B^0)k^1 \qquad (3.21)
\end{aligned}
$$

Differently from the Quadratic Identity established by Corollary (2.2), Corollary (3.2) is not an "if and only if" result: the decomposition of value difference between two quadratic functions imply (3.18)-(3.20), but the converse is not true if they have different parameters. The equations (3.18)-(3.20) may hold, as a particular case, also with functional forms that are linear in z. Note that the inequality $\nabla_z f^{\,0}(z^0) \neq \nabla_z f^{\,1}(z^1)$ for $z^0 \neq z^1$ may occur with linear functions when these have different parameters.

Furthermore, we observe that the theory of exact and superlative index numbers has traditionally considered quadratic functional forms where only the parameters of the zero- and first-order terms in z may differ. By contrast, the result obtained by Corollary (3.2) shows that, also when the parameters of second-order terms are different, the difference in functional values of two quadratic functions can be still split into two separate components due to differences in z and differences in parameters.

LEMMA 3.2. Quadratic Approximation of a Value Difference of Two Arbitrary Functions with Different Parameters or Functional Forms. *Let* $f^1(z) \equiv h^1(z, k^1)$ and $f^0(z) \equiv h^0(z, k^0)$ *be two arbitrary once differentiable functions with different parameters or functional forms. If their value difference is accounted for by using (3.5) where* θ *is set equal to 1/2 as in (3.16), then , for all* z^0 *and* z^1,

$$
\begin{aligned}
f^1(z^1) - f^0(z^0) \; = \; & \frac{1}{2}[\nabla_z f^0(z^0) + \nabla_z f^1(z^1)](z^1 - z^0) \\
& + \textit{Parameter change component} \\
& + \textit{Error of approximation} \qquad (3.22)
\end{aligned}
$$

where

$$\begin{aligned} & \textit{Parameter change component} \\ \equiv \; & \theta[f^1(z^0) - f^0(z^0)] + (1-\theta)[f^1(z^1) - f^0(z^1)] \end{aligned} \tag{3.23}$$

and

$$\textit{Error of approximation} \equiv \left(\frac{1}{2} - \theta\right) \left\{ \left[f_L^0(z^1) - f_L^0(z^0)\right] - \left[f_L^1(z^1) - f_L^1(z^0)\right] \right\} \tag{3.24}$$

with $\theta = \theta^*(z^0, z^1) \equiv \frac{R_1^0(z^0,z^1)}{R_1^0(z^0,z^1)+R_1^1(z^0,z^1)}$ *if* $R_1^0(z^0, z^1) + R_1^1(z^0, z^1) \neq 0$, *or* θ *may take any value if* $R_1^0(z^0, z^1) = R_1^1(z^0, z^1) = 0$, *and* $f_L^t(z)$ *is a first-order (linear) approximating function that is tangent to* $f^t(z)$ *at* z^t *(that is* $f_L^t(z) \equiv a^t + b^t z = f^t(z^t) + \nabla_z f^t(z^t)(z - z^t)$, *with* $a^t = f^t(x^t) - \nabla_z f^t(x^t) x^t$ *and* $b^t = \nabla_z f^t(x^t)$, *with* $t = 0, 1$*).*

Let us, now, construct the following general decomposition of the absolute value difference between $h^1(z^1, k^1)$ and $h^0(z^0, k^0)$:

$$h^1(z^1, k^1) - h^0(z^0, k^0)$$

$$\begin{aligned} &= (1-\lambda)\ [h^0(z^1, k^0) - h^0(z^0, k^0)] + \lambda\ [h^1(z^1, k^1) - h^1(z^0, k^1)] \\ &\quad + \lambda\ [h^1(z^0, k^1) - h^0(z^0, k^0)] + (1-\lambda)\ [h^1(z^1, k^1) - h^0(z^1, k^0)] \end{aligned} \tag{3.25}$$

The following technical result is also obtained:

COROLLARY 3.3. Accounting for the Sum of Value Differences between Two Quadratic Functions with Different "Zero-order" and "First-order" Parameters (Caves, Christensen, and Diewert's, 1982, pp.1412-1413 *Translog Identity*). *If two quadratic function defined by (3.12) have different parameters of the "zero-order" and "first-order" terms but equal parameters the same "second-order" terms in z (that is* $A^1 = A^0 = A$*), then, for all* z^0, z^1, k^0, k^1,

$$\left[f_Q^0(z^1) - f_Q^0(z^0)\right] + \left[f_Q^1(z^1) - f_Q^1(z^0)\right] \tag{3.26}$$

or, equivalently,

$$\left[h_Q^0(z^1, k^0) - h_Q^0(z^0, k^0)\right] + \left[h_Q^1(z^1, k^1) - h_Q^1(z^0, k^1)\right]$$

$$= \left[\nabla_z h_Q^0(z^0, k^0) + \nabla_z h_Q^1(z^1, k^1)\right]^T (z^1 - z^0) \tag{3.27}$$

This result is known under the name of '*Translog' Identity*, because it was formulated by Caves, Christensen, and Diewert (1982, pp. 1412-1413) in terms of translog functions.

In the more general case of quadratic functions with no restrictions on variable parameters, we have the following result:

COROLLARY 3.4. Accounting for the Sum of Value Differences between Two Quadratic Functions with Different Parameters (Caves, Christensen, and Diewert, 1982, pp.1412-1413). *If two quadratic functions are defined by (3.11)-(3.12), differing in all parameters, including those of the "second-order" terms in z (A^1 may differ from A^0), then, for all z^0, z^1, k^0, k^1,*

$$\left[f_Q^0(z^1) - f_Q^0(z^0)\right] + \left[f_Q^1(z^1) - f_Q^1(z^0)\right] \tag{3.28}$$

or, equivalently,

$$\left[h_Q^0(z^1, k^0) - h_Q^0(z^0, k^0)\right] + \left[h_Q^1(z^1, k^1) - h_Q^1(z^0, k^1)\right]$$

$$\begin{aligned} = \; & \left[\nabla_z h_Q^0(z^0, k^0) + \nabla_z h_Q^1(z^1, k^1)\right] (z^1 - z^0) \\ & -\frac{1}{2}(z^1 - z^0)(A^1 - A^0)(z^1 - z^0) \end{aligned} \tag{3.29}$$

If $A^1 = A^0$, then the term $\frac{1}{2}(z^1 - z^0)(A^1 - A^0)(z^1 - z^0)$ is equal to zero and the result obtained with Corollary (3.4) is the same of that obtained with Corollary (3.3). It is evident that Caves, Christensen, and Diewert (1982, pp. 1412-1413) introduced the restriction $A^1 = A^0$ in order to obtain the equivalence between an arithmetic average of the two functions (evaluated at z^0 and z^1) and the component of weighted changes in z. This equivalence no longer holds when $A^1 \neq A^0$, but the following result is useful to show that, even in this general case, the overall functional value difference can be still decomposed into separate components of differences in z, k, and variable parameters.

COROLLARY 3.5: *If two functions* $f^t_Q(z) \equiv h^t_Q(z, k^t)$ *are defined by (3.11)-(3.12), then, for all* z^0, z^1, k^0, k^1,

$$
\begin{aligned}
& \left[f^1_Q(z^0) - f^0_Q(z^0)\right] + \left[f^1_Q(z^1) - f^0_Q(z^1)\right] \\
= \;& (a^1 - a^0)^T(z^0 + z^1) + 2(a^1_0 - a^0_0) \\
& + z^0(A^1 - A^0)z^1 + \frac{1}{2}(z^1 - z^0)(A^1 - A^0)(z^1 - z^0) \qquad (3.30)
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
& \left[h^1_Q(z^0, k^1) - h^0_Q(z^0, k^0)\right] + \left[h^1_Q(z^1, k^1) - h^0_Q(z^1, k^0)\right] \\
= \;& \left[h^1_Q(z^1, k^1) - h^1_Q(z^1, k^0)\right] + \left[h^0_Q(z^0, k^1) - h^0_Q(z^0, k^0)\right] \\
& + \left[h^1_Q(z^1, k^1) - h^0_Q(z^1, k^1)\right] + \left[h^1_Q(z^0, k^0) - h^0_Q(z^0, k^0)\right] \qquad (3.31)
\end{aligned}
$$

where

$$
\begin{aligned}
& \left[h^1_Q(z^1, k^1) - h^1_Q(z^1, k^0)\right] + \left[h^0_Q(z^0, k^1) - h^0_Q(z^0, k^0)\right] \\
= \;& [\nabla_k h^0_Q(z^0, k^0) + \nabla_k h^1_Q(z^1, k^1)]^T(k^1 - k^0) \\
& - \frac{1}{2}(k^1 - k^0)(B^1 - B^0)(k^1 - k^0) \qquad (3.32)
\end{aligned}
$$

and

$$
\begin{aligned}
& \left[h^1_Q(z^1, k^1) - h^0_Q(z^1, k^1)\right] + \left[h^1_Q(z^0, k^0) - h^0_Q(z^0, k^0)\right] \\
= \;& 2(\alpha^1_0 - \alpha^0_0) + (\alpha^1 - \alpha^0)^T(z^1 + z^0) \\
& + z^0(A^1 - A^0)z^1 + \frac{1}{2}(z^1 - z^0)(A^1 - A^0)(z^1 - z^0) \\
& + (k^1 + k^0)^T(\beta^1 - \beta^0) \\
& + k^0(B^1 - B^0)k^1 + \frac{1}{2}(k^1 - k^0)(B^1 - B^0)(k^1 - k^0) \qquad (3.33)
\end{aligned}
$$

Adding (3.29) to (3.31)-(3.33) and dividing the resulting expression through by 2 yield the same result obtained by Corollary (3.2).

It is straightforward to show that, in order to obtain $\frac{1}{2}\left[\nabla_z h_Q^0(z^0,k^0)\ +\nabla_z h_Q^1(z^1,k^1)\right]^T$ $\cdot(z^1-z^0)=(1-\lambda)\ [f^0(z^1)\ -\ f^0(z^0)]+\lambda[f^1(z^1)\ -\ f^1(z^0)],\ \ \lambda$ must take the particular value

$$\lambda^*(z^0,z^1)\quad\equiv\quad\frac{\frac{1}{2}\left(F^1-F^0\right)+\frac{1}{4}(z^1-z^0)(A^1-A^0)(z^1-z^0)}{(F^1-F^0)}\qquad(3.34)$$

where $F^t\equiv[f^t(z^1)-f^t(z^0)]$ for $t=0,1$. If $A^1=A^0$, then $\lambda^*(z^0,z^1)=1/2)$. This is the case traditionally considered in this context. If $A^1=A^0$ and $B^1=B^0$, then equations (3.20) and (3.21) reduce to the decomposition procedure used, for example, by Diewert and Morrison (1986, Theorem 1, p. 663) in terms of translog functions with variable "zero-order" and "first-order" parameters and constant "second-order" parameters.

From our results, it is evident that in the more general case, where $A^1\neq A^0$, it is still possible to decompose the difference between two quadratic functions defined by (3.11) into a component due to (weighted) changes in z and a component due to (weighted) differences in parameters. These two components turn out to be a weighted average of two functional value changes evaluated at the respective reference variables, with weights that may differ from 1/2. Similarly, it is still possible to decompose the functional value changes in $h_Q^t(z,k)$ defined by (3.12) into three separate components, respectively due to (weighted) changes in z, k, and parameters. This remarkable result widens considerably the scope of applicability of the decomposition of differences in functional values of quadratic functions.

COROLLARY 3.6. Accounting for Functional Value Ratios of Two Different Quadratic Homothetic Functions (Byushgens, 1925; Konüs and Byushgens, 1926; Frisch, 1936, p. 30; Wald, 1939, p. 331; Pollak, 1971; Afriat, 1972, p. 45; 1977, pp. 141-143; 2005, pp. 177-178). *If two continuously once-differentiable quadratic functions $f_Q^0(z)$ and $f_Q^1(z)$ defined by (3.11) with different parameters are homothetic transformation of a linearly homogeneous function*[14]*, so that $a_i=0$ for $i=0,1,...,N$, and $f_Q^t(z)=\sigma^t[\phi(z)]^2$ for $t=0,1$, where $\phi(z)\equiv[\frac{1}{2}z^TAz]^{1/2}$, then, for all z^0 and z^1,*

$$\frac{f_Q^1(z^1)}{f_Q^0(z^0)}\quad=\quad I_Z\cdot I_Y\cdot I_T\qquad(3.35)$$

[14] In a linearly homogeneous function, if all the variables change proportionally, also the resulting functional value changes by the same factor of proportionality.

where

$$I_Y \quad = \quad I_Z \equiv \left[\frac{\sum_{i=1}^N s_{Q,i}^0 \frac{z_i^1}{z_i^0}}{\sum_{i=1}^N s_{Q,i}^1 \frac{z_i^0}{z_i^1}} \right]^{1/2} \qquad (3.36)$$

with $s_{Q,i}^t \equiv \frac{\partial f_Q^t(z^t)}{\partial z_i^t} \cdot z_i^t / \sum_{i=1}^N \frac{\partial f_Q^t(z^t)}{\partial z_i^t} \cdot z_i^t$ *(*$t = 0, 1$*),*

$$I_T \quad \equiv \quad \frac{\sigma^1}{\sigma^0} \qquad (3.37)$$

We note that (3.36) is an ideal Fisher-type index number, which is "exact" for (identically equal to) the ratio $\phi(z^1)/\phi(z^0) = [\frac{1}{2} z^{1T} A z^1]^{1/2} / [\frac{1}{2} z^{0T} A z^0]^{1/2}$.

We finally observe that an alternative procedure can be set up using Divisia indexes in order to account for value changes in each of the two functions taken into exam here. This procedure should lead us to the same results that we have obtained in this section.

4. *Accounting for value changes of transformed functions*

In the previous sections, we have seen that the Quadratic Identity, in its original formulation, is useful only in the very restrictive case of quadratic functions and cannot be applied to arbitrary functions without incurring a possible non-negligible error. In the general case, a way to reduce this error is to use a transformed quadratic function. With appropriate parameter values, it is this function rather than a quadratic function that provides a second-order differential approximation to an arbitrary function.

Suppose that the arbitrary function $f^t(z)$ considered in section 3 can be defined as $f^t[Z(q)]$, where $Z(q) = z$, with the i^{th} element $z_i = z(q_i)$, so that $f^t[Z(q)] = f^t(q)$. Furthermore, suppose that a general quadratic function $f_{GQ}^t(q)$ can be transformed into a quadratic function as follows:

$$g\left[f_{GQ}^t(q)\right] \quad = \quad a_0^t + a^t Z^t(q) + \frac{1}{2} Z^t(q) A^t Z^t(q) \qquad (4.1)$$

where $Z^t(q) \equiv [z^t(q_1)\ z^t(q_2)\ ...\ z^t(q_N)]^T$, so that

$$f^t_{GQ}(q) \quad = \quad g^{-1}\left[a^t_0 + a^t Z^t(q) + \frac{1}{2}Z^t(q)A^t Z^t(q)\right] \qquad (4.2)$$

Let us define $a^t_0 \equiv \alpha^t_0 + \beta^t K(x^t) + \frac{1}{2}K(x^t)B^t K(x^t)$ and $a^t \equiv \alpha^t + K(x^t)\Gamma^t$ where $K(x) \equiv [k(x_1)\ k(x_2)\ ...\ k(x_M)]^T$ so that $g\left[f^t_{GQ}(q)\right] \equiv g\left[h^t_{GQ}(q, x^t)\right]$, with

$$\begin{aligned} g\left[h^t_{GQ}(q, x)\right] \quad = \quad & \alpha^t_0 + \alpha^t_i Z(q) + \frac{1}{2}Z(q)A^t Z(q) \\ & +K(x)\beta^t + \frac{1}{2}K(x)B^t K(x) \\ & +K(x)\Gamma^t Z(q) \end{aligned} \qquad (4.3)$$

where $z_i \equiv z(q_i)$ and $k_m \equiv K(x_m)$, or in vector notation $z \equiv Z(q)$ and $k \equiv K(x)$, so that

$$\begin{aligned} h^t_{GQ}(q, x) \quad = \quad & g^{-1}[\alpha^t_0 + \alpha^t_i Z(q) + \frac{1}{2}Z(q)A^t Z(q) \\ & +K(x)\beta^t + \frac{1}{2}K(x)B^t K(x) \\ & +K(x)\Gamma^t Z(q)] \end{aligned} \qquad (4.4)$$

The values of all parameters may change as t changes, and g, z, and k are continuous and monotonic functions of one single variable with non-zero derivatives.

Since the functions z and k are continuous and monotonic, it is possible to invert them in order to obtain

$$q_i = z^{-1}(z_i) \text{ or, in vector form, } \ q = Z^{-1}(z) \qquad (4.5)$$

$$x_m = k^{-1}(k_m) \text{ or, in vector form, } \ x = K^{-1}(k) \qquad (4.6)$$

so that

$$\begin{aligned} g[f^t_{GQ}(q)] \quad & = \quad g\left\{f^t_{GQ}[Z^{-1}(z)]\right\} \\ & \equiv \quad f^t_Q(z) \quad \text{defined by (3.11)} \end{aligned} \qquad (4.7)$$

and

$$\begin{aligned} g[h^t_{GQ}(q,x)] &= g\left\{h^t_{GQ}[Z^{-1}(z), K^{-1}(k)]\right\} \\ &\equiv h^t_Q(z,k) \quad \text{defined by (3.12)} \end{aligned} \tag{4.8}$$

If the functions f^t_Q and h^t_Q have specific parameter values such that $f^t_{GQ}(q^*) = f^t(q^*)$, $\nabla f^t_{GQ}(q^*) = \nabla f^t(q^*)$, and $\nabla^2 f^t_{GQ}(q^*) = \nabla^2 f^t(q^*)$, then the function $f^t_{GQ} = g^{-1}(f^t_Q)$ provides a second-order differential approximation to f^t around q^*. This should be contrasted with the assumption in section 3 that it is f^t_Q rather than $g^{-1}(f^t_Q)$ that provides a second-order differential approximation to f^t around q^*.

We can now obtain the following result:

LEMMA 4.1. Accounting for Value Differences of Two General Quadratic Functions. *If the function $f^t_{GQ}(q)$ is defined by (4.2), then*

$$\begin{aligned} & g[f^1_{GQ}(q^1)] - g[f^0_{GQ}(q^0)] \\ =\; & \frac{1}{2}\left\{g'[f^0_{GQ}(q^0)]\cdot[\widehat{Z}'(q^0)]^{-1}\cdot\nabla_q f^0_{GQ}(q^0) + g'[f^1_{GQ}(q^1)]\cdot[\widehat{Z}'(q^1)]^{-1}\cdot\nabla_q f^1_{GQ}(q^1)\right\}^T \\ & \cdot[(Z(q^1) - Z(q^0)] + T \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} T \equiv\; & (a^1_0 - a^0_0) + (a^1 - a^0)\frac{1}{2}[Z(q^0) + Z(q^1] \\ & + Z(q^0)(A^1 - A^0)Z(q^0) \end{aligned} \tag{4.10}$$

$[\widehat{Z}'(q)]^{-1}$ *is a diagonal matrix in which the $(i,i)^{th}$ element is equal to*

$$\frac{dq_i}{dz_i} = \frac{dz^{-1}(z_i)}{dz_i} = 1/z'(q_i) \tag{4.11}$$

(with $z'(q_i) \neq 0$, by assumption).

5. Full specification of a general transformed quadratic function

The transformed quadratic function (4.1)-(4.2) can be fully specified by choosing the functional form of the transformation functions g, z, and k. Among the many candidates, if we define the following functions as suggested in the original work of Box and Cox (1964):

$$g(y) \equiv \frac{y^{\rho}-1}{\rho} \tag{5.1}$$

$$z(q_i) \equiv \frac{q_i^{\lambda}-1}{\lambda} \tag{5.2}$$

$$k(x_i) \equiv \frac{x_i^{\lambda}-1}{\lambda} \tag{5.3}$$

and replace the functions (5.1) and (5.2) in (4.1), then we obtain the following quadratic Box-Cox function[15]:

$$\frac{\left[f^t_{Q^{\rho,\lambda}}(q)\right]^{\rho}-1}{\rho} = a^t_0 + \sum_{i=1}^{N} a^t_i \frac{q_i^{\lambda}-1}{\lambda} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a^t_{ij}\frac{q_i^{\lambda}-1}{\lambda}\frac{q_j^{\lambda}-1}{\lambda} \tag{5.4}$$

which, by replacing (5.1)-(5.3) in (4.3), corresponds to

$$\begin{aligned}\frac{\left[h^t_{Q^{\rho,\lambda}}(q,x)\right]^{\rho}-1}{\rho} = {} & \alpha^t_0 + \sum_{i=1}^{N} \alpha^t_i \frac{q_i^{\lambda}-1}{\lambda} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a^t_{ij}\frac{q_i^{\lambda}-1}{\lambda}\frac{q_j^{\lambda}-1}{\lambda} \\ & + \sum_{m=1}^{M} \beta^t_m \frac{x_m^{\lambda}-1}{\lambda} + \frac{1}{2}\sum_{m=1}^{M}\sum_{n=1}^{M} b^t_{mn}\frac{x_m^{\lambda}-1}{\lambda}\frac{x_n^{\lambda}-1}{\lambda} \\ & \sum_{i=1}^{N}\sum_{m=1}^{M} \gamma^t_{mi}\frac{x_m^{\lambda}-1}{\lambda}\frac{q_i^{\lambda}-1}{\lambda}\end{aligned} \tag{5.5}$$

[15] The explicit use of the quadratic Box-Cox function can be dated back at least to the works of Khaled (1977), Kiefer (1977), Appelbaum (1979), and Berndt and Khaled (1979). The special case of a quadratic Box-Cox aggregator function has also been derived by Diewert (1980, pp. 450-451) starting from a quadratic mean-of-order-r aggregator function.

by setting

$$a_0^t \equiv \alpha_0^t + \sum_{m=1}^{M} \beta_m^t \frac{x_m^\lambda - 1}{\lambda} + \frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{M} b_{mn}^t \frac{x_m^\lambda - 1}{\lambda} \frac{x_n^\lambda - 1}{\lambda} \tag{5.6}$$

$$a_i^t \equiv \alpha_i^t + \sum_{m=1}^{M} \gamma_{mi}^t \frac{x_m^\lambda - 1}{\lambda} \tag{5.7}$$

When $\rho \to 0$ and $\lambda \to 0$, the function (5.4) reduces to the following translog functional form:

$$\ln f_{Trg}^t(q) = a_0^t + \sum_{i=1}^{N} a_i^t \ln q_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^t \ln q_i \ln q_j \tag{5.8}$$

so that

$$f_{Trg}^t(q) \equiv \exp a_0^t \cdot \exp(\sum_{i=1}^{N} a_i^t \ln q_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^t \ln q_i \ln q_j) \tag{5.9}$$

or, equivalently, the function (5.5) reduces to

$$\begin{aligned} \ln h_{Trg}^t(q,x) = {} & \alpha_0^t + \sum_{i=1}^{N} \alpha_i^t \ln q_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^t \ln q_i \ln q_j \\ & + \sum_{m=1}^{M} \beta_m^t \ln x_m + \frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{M} b_{mn}^t \ln x_m \ln x_n \\ & + \sum_{m=1}^{M} \sum_{i=1}^{N} \gamma_{mi}^t \ln x_m \ln q_i \end{aligned} \tag{5.10}$$

so that

$$\begin{aligned} h_{Trg}^t(q,x) \equiv {} & \exp \alpha_0^t \cdot \exp(\sum_{i=1}^{N} \alpha_i^t \ln q_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^t \ln q_i \ln q_j) \\ & \cdot \exp(\sum_{m=1}^{M} \beta_m^t \ln x_m + \frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{M} b_{mn}^t \ln x_m \ln x_n) \\ & \cdot \exp(\sum_{m=1}^{M} \sum_{i=1}^{N} \gamma_{mi}^t \ln x_m \ln q_i) \end{aligned} \tag{5.11}$$

The function f_{Trg}^t is homogeneous of degree one in q under the following conditions:

(*i*) $\sum_{i=1}^{N} \alpha_i^t = 1$

(*ii*) $\sum_{i=1}^{N} a_{ij}^t = 0$ for $j = 1, 2, \ldots, N$,

$(iii) \sum_{j=1}^{N} a_{ij}^{t} = 0, \quad$ for $i = 1,2,\ldots, N$
(condition (ii) implies (iii) under the symmetry of matrix A), and
$(iv) \sum_{i=1}^{N} \gamma_{mi} = 0$, for $m = 1,\ldots,M$

Similarly, the function h_{Trg}^{t} is homogeneous of degree one in x under the following conditions:
$(v) \sum_{m=1}^{M} \beta_{m}^{t} = 1$,
$(vi) \sum_{m=1}^{M} b_{mn} = 0$, for $j = 1,2,\ldots, N$,
$(vii) \sum_{n=1}^{M} b_{mn} = 0$, for $i = 1,2,\ldots, N$
(condition (vi) implies (vii) under the symmetry of matrix B), and
$(viii) \sum_{m=1}^{M} \gamma_{mi} = 0$, for $n = 1,2,\ldots, N$.

When $\rho \neq 0$ and $\lambda \neq 0$, then the function (5.4) has the following functional form:

$$f_{Q^{\rho,\lambda}}^{t}(q) = \left[\overline{a}_0^t + \sum_{i=1}^{N} \overline{a}_i^t q_i^{\lambda} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \overline{a}_{ij}^t q_i^{\lambda} q_j^{\lambda}\right]^{\frac{1}{\rho}} \left(\frac{1}{\lambda}\right)^{\frac{1}{\rho}} \quad (5.12)$$

where $\overline{a}_0^t \equiv \left(\lambda + \rho\lambda a_0^t - \rho \sum_{i=1}^{N} a_i^t + \frac{1}{2}\frac{\rho}{\lambda} \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^t\right)$, $\overline{a}_i^t \equiv \left(\rho a_i^t - \frac{\rho}{\lambda}\sum_{j=1}^{N} a_{ij}^t\right)$, and $\overline{a}_{ij}^t \equiv \frac{\rho}{\lambda} a_{ij}^t$. The function $f_{Q^{\rho,\lambda}}^{t}$ is homogeneous of degree $2\lambda/\rho$ in q if $\overline{a}_0^t = 0$ and $\overline{a}_i^t = 0$ for all i's, that is if the following conditions are satisfied:

$$(1 + \rho a_0^t) = \frac{1}{2}\frac{\rho}{\lambda}\sum_{i=1}^{N} a_i^t \quad (5.13)$$

$$\lambda a_i^t = \sum_{j=1}^{N} a_{ij}^t \quad (5.14)$$

These two conditions imply

$$\frac{\rho}{2\lambda^2} = \frac{(1 + \rho a_0^t)}{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^t} \quad (5.15)$$

In this case, the function $f_{Q^{\rho,\lambda}}^{t}$ is homogeneous of degree $2\lambda/\rho$ in q and can be re-expressed as

$$
\begin{aligned}
f^t_{Q^{\rho,\lambda}}(q) &= \left[\frac{1}{2}\frac{r}{\lambda}\sum_{i=1}^{N}\sum_{j=1}^{N} a^t_{ij} q_i^{\lambda} q_j^{\lambda}\right]^{\frac{1}{\rho}} \left(\frac{1}{\lambda}\right)^{\frac{1}{\rho}} \\
&= \left[\sum_{i=1}^{N}\sum_{j=1}^{N} a^t_{ij} q_i^{\lambda} q_j^{\lambda}\right]^{\frac{1}{\rho}} \left[\frac{1+ra^t_0}{\sum_{i=1}^{N}\sum_{j=1}^{N} a^t_{ij}}\right]^{\frac{1}{\rho}} \qquad (5.16)
\end{aligned}
$$

since $\frac{1}{\lambda} = \frac{2\lambda}{\rho}(1+\rho a^t_0)/\sum_{i=1}^{N}\sum_{j=1}^{N} a^t_{ij}$ because of the restrictions (5.13) and (5.14) or (5.15) for the homogeneity of degree $2\lambda/\rho$ of $f^t_{Q^{\rho,\lambda}}$. Hence

$$
f^t_{Q^{\rho,\lambda}}(q) = (1+\rho a^t_0)^{\frac{1}{r}} \left[\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha^t_{ij} q_i^{\lambda} q_j^{\lambda}\right]^{\frac{1}{\rho}} \qquad (5.17)
$$

where $\alpha^t_{ij} \equiv \frac{a^t_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{N} a^t_{ij}}$. We note that $\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha^t_{ij} q_i^{\lambda} q_j^{\lambda} = 1$ when all q_i's are equal to 1.

Moreover, under the degree $2\lambda/\rho$ homogeneity conditions (5.13)-(5.14), if all parameters a^t_{ij} in (5.16) change proportionally, so that α^t_{ij} in (5.17) remain constant ($\alpha^t_{ij} = \alpha_{ij}$), then the effects of parameter changes are separable from q. This function can be seen as a homothetic transformation of a linearly homogeneous function, that is

$$
f^t_{Q^{\rho,\lambda}}(q) = \sigma^t [f_{Q^r}(q)]^{r/\rho} \qquad (5.18)
$$

where $r = 2\lambda$, $\sigma^t \equiv (1+\rho a^t_0)^{\frac{1}{\rho}}$, and

$$
f_{Q^r}(q) \equiv \left[\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_{ij} q_i^{\frac{r}{2}} q_j^{\frac{r}{2}}\right]^{\frac{1}{r}} \qquad (5.19)
$$

which is the *quadratic mean-of-order-r* aggregator function used by Diewert's (1976, pp. 129-130)[16]. This function is separable from parameter changes and homogenous of degree one in q.

[16] This functional form is due to McCarthy (1967), Kadiyala (1972), Denny (1972, 1974), and Hasenkamp (1973).

Equation (5.19) reduces to well-known functions for particular values of r. Denny (1972, 1974) noted that, if $r = 1$, then it reduces to the generalized linear functional form proposed by Diewert (1969, 1971). In an unpublished memorandum, Lau (1973) showed that, at the limit as r tends to zero, it reduces to the homogeneous translog aggregator function (Lau's proof is reported in Diewert, 1980, p. 451). Diewert (1976, p. 130) also noted that, if $r = 2$, then it reduces to the Konüs-Byushgens (1926) functional form. Furthermore, if all $\alpha_{ij} = 0$ for $i \neq j$, then it reduces to a CES functional form.

If the parameters of the first-order terms of the function (5.12) are functions of x as follows

$$\overline{a}_0^t \equiv \overline{\alpha}_0^t + \sum_{m=1}^{M} \overline{\beta}_m^t x_m^\lambda + \frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{M} b_{mn}^t x_m^\lambda x_n^\lambda \tag{5.20}$$

$$\overline{a}_i^t \equiv \overline{\alpha}_i^t + \sum_{m=1}^{M} \gamma_{mi}^t x_m^\lambda, \tag{5.21}$$

then (5.12) is identically equal to

$$\begin{aligned} h_{Q^{\rho,\lambda}}^t(q,x) = {} & [\overline{\alpha}_0^t + \sum_{i=1}^{N} \overline{\alpha}_i^t q_i^\lambda + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^t q_i^\lambda q_j^\lambda \\ & + \sum_{m=1}^{M} \overline{\beta}_m^t x_m^\lambda + \frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{M} b_{mn}^t x_m^\lambda x_n^\lambda \\ & + \sum_{i=1}^{N} \sum_{m=1}^{M} \gamma_{mi}^t x_m^\lambda q_i^\lambda]^{\frac{1}{\rho}} \left(\frac{1}{\lambda}\right)^{\frac{1}{\rho}} \end{aligned} \tag{5.22}$$

where

$\overline{\alpha}_0^t \equiv (\lambda + \rho\lambda\alpha_0^t - \rho \sum_{i=1}^{N} \alpha_i^t + \frac{1}{2}\frac{\rho}{\lambda} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^t - \rho \sum_{m=1}^{M} \beta_m^t + \frac{1}{2}\frac{\rho}{\lambda} \sum_{m=1}^{M} \sum_{n=1}^{M} b_{mn}^t$
$+ \frac{\rho}{\lambda} \sum_{m=1}^{M} \sum_{i=1}^{N} \gamma_{mi}^t)$;

$\overline{\alpha}_i^t \equiv (\rho\alpha_i^t - \frac{\rho}{\lambda} \sum_{j=1}^{N} a_{ij}^t - \frac{\rho}{\lambda} \sum_{m=1}^{M} \gamma_{mi}^t)$;

$\overline{\beta}_m^t \equiv (\rho\beta_m^t - \frac{\rho}{\lambda} \sum_{n=1}^{M} b_{mn}^t - \frac{\rho}{\lambda} \sum_{i=1}^{N} \gamma_{mi}^t)$

Under the parameter restrictions (5.13) and (5.14) imposed on (5.12) for the homogeneity of degree $2\lambda/\rho$ in q, using the definitions (5.6) and (5.7), equation (5.22) becomes

$$
\begin{aligned}
h^t_{Q^{\rho,\lambda}}(q,x) &= \left[\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha^t_{ij}q_i^\lambda q_j^\lambda\right]^{\frac{1}{\rho}} \\
&\cdot\left[\overline{\beta}^t_0+\sum_{m=1}^{M}\overline{\beta}^{*t}_m x^\lambda_m+\frac{1}{2}\sum_{m=1}^{M}\sum_{n=1}^{M}\overline{b}^t_{mn}x^\lambda_m x^\lambda_n\right]^{\frac{1}{\rho}}\left(\frac{1}{\lambda}\right)^{\frac{1}{\rho}} \quad (5.23)
\end{aligned}
$$

where $\overline{\beta}^t_0 \equiv (\lambda + \rho\lambda\alpha^t_0 - \rho \sum_{m=1}^{M} \beta^t_m + \frac{1}{2}\frac{\rho}{\lambda}\sum_{m=1}^{M}\sum_{n=1}^{M} b^t_{mn})$;

$\overline{\beta}^{*t}_m \equiv (\rho\beta^t_m - \frac{\rho}{\lambda}\sum_{n=1}^{M} b^t_{mn}) \equiv \overline{\beta}^t_m + \frac{\rho}{\lambda}\sum_{i=1}^{N}\gamma^t_{mi} = \overline{\beta}^t_m$ since $\sum_{i=1}^{N}\gamma^t_{mi} = 0$ as a condition imposed for the homogeneity of degree $2\lambda/\rho$ in q;

$\overline{b}^t_{mn} = \frac{\rho}{\lambda}b^t_{mn}$.

Moreover, if

$$
\begin{aligned}
(1+\rho\alpha^t_0) &= \frac{1}{2}\frac{\rho}{\lambda}\sum_{m=1}^{M}\beta^t_m \quad (5.24)\\
\lambda\beta^t_m &= \sum_{n=1}^{M} b^t_{mn} \quad (5.25)\\
\sum_{m=1}^{N}\gamma^t_{mi} &= 0, \quad (5.26)
\end{aligned}
$$

then $\overline{\beta}^t_0 = \overline{\beta}^{*t}_m = 0$ (and, therefore, $\overline{\beta}^t_m = 0$), and $h^t_{Q^{\rho,\lambda}}$ is homogeneous of degree $2\lambda/\rho$ in q as well as x. In this case, (5.23) reduces to

$$
\begin{aligned}
h^t_{Q^{\rho,\lambda}}(q,x) &= (1+\rho\alpha^t_0)^{\frac{1}{\rho}}\cdot\left[\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha^t_{ij}q_i^{\frac{r}{2}}q_j^{\frac{r}{2}}\right)^{\frac{1}{r}}\right]^{\frac{r}{\rho}} \\
&\cdot\left[\left(\sum_{m=1}^{M}\sum_{n=1}^{M}\beta^t_{mn}x_m^{\frac{r}{2}}x_n^{\frac{r}{2}}\right)^{\frac{1}{r}}\right]^{\frac{r}{\rho}} \quad \text{with } r = 2\lambda \quad (5.27)
\end{aligned}
$$

where $\beta^t_{ij} \equiv \frac{b^t_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{N} b^t_{ij}}$. Note, that $\sum_{m=1}^{M}\sum_{n=1}^{M}\beta^t_{mn}x_m^{\frac{r}{2}}x_n^{\frac{r}{2}} = 1$ if all x_i's and x_j's are equal to 1[17].

Note that, with α^t_{ij} and β^t_{mn} constant over t, for all i, j, m, and n, the function $h^t_{Q^{\rho,\lambda}}(q,x)$ is homothetically separable in q, x, and from parameter

[17] We may note that, with $r = \rho = 2$ and all "second-order" parameters being constant ($\alpha^t_{ij} = \alpha_{ij}$ and $\beta^t_{mn} = \beta_{mn}$ for every value of t), (5.27) reduces to the quadratic mean-of-order-2 functional form used by Diewert (1992, p. 231, eq. (56)).

changes. The last ones are captured by changes in the factor of proportionality $(1+\rho\alpha_0^t)^{\frac{1}{\rho}}$. The resulting homogeneous functions $\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_{ij}q_i^{\frac{r}{2}}q_j^{\frac{r}{2}}\right]^{\frac{1}{r}}$ and $\left[\sum_{m=1}^{M}\sum_{n=1}^{M}\beta_{mn}x_m^{\frac{r}{2}}x_n^{\frac{r}{2}}\right]^{\frac{1}{r}}$ can be considered as "aggregators" of q and x, respectively[18].

6. Accounting for value changes of translog and quadratic mean-of-order-r functional forms

The general quadratic mean-of-order-r function, which is algebraically derivable from the quadratic Box-Cox function defined by (5.4)-(5.5), can be used as a second-order approximation to the an arbitrary unknown function. Using the results obtained thus far, it is possible to account for differences in functional values, either in terms of differences or in terms of ratios, into aggregating components of changes in the arguments and parameters of the function. We shall follow the tradition of calling *indicators* the aggregating components defined in terms of differences and *index numbers* those that are defined in terms of ratios (see, for example, Diewert, 1998, 2000). In order to save space, we shall deal with only the function $f^t_{Q^{\rho,\lambda}}(q)$ defined by (5.12) and leave to the reader the exercise of deriving the corresponding results with the more explicit function $h^t_{Q^{\rho,\lambda}}(q,x)$ defined by (5.22). The following theorems are in order.

THEOREM 6.1. *If the general quadratic Box-Cox function with variable parameters defined by (5.4) reduces to the "translog" function (5.6), with* $\rho \to 0$ *and* $\lambda \to 0$, *then*

$$\ln f^1_{Trg} - \ln f^0_{Trg} = \frac{1}{2}\sum_{i=1}^{N}\left(s^0_{Trg,i}+s^1_{Trg,i}\right)\left(\ln q^1_i - \ln q^0_i\right) + \textit{Parameter-change component} \qquad (6.1)$$

where

$$s^t_{Trg,i} \equiv \frac{q^t_i f^t_{Trgi}}{f^t_{Trg}} \text{ with } t=0,1 \text{ and } f^t_{Trgi} \equiv \frac{\partial f^t_{Trg}}{\partial q^t_i} \qquad (6.2)$$

[18] In section 3, we have recalled that an aggregator function must be degree-one homogeneous as well as separable.

We note that, in general, $s^t_{Trg,i} \neq s^t_i$, where $s^t_i \equiv q^t_i f^t_i / f^t$ and $f^t_i \equiv \partial f^t / \partial q^t_i$, except at the point of approximation. Moreover, $\sum_{i=1}^N s^t_{Trg,i} = 1$ if f^0_{Trg} and f^1_{Trg} are homogeneous of degree one in q, since in this case $f^t_{Trg} = \sum_{i=1}^N q^t_i f^t_{Trg,i}$ by Euler's theorem. If $\sum_{i=1}^N s^t_{Trg,i} \neq 1$, then we can define $s^{*t}_{Trg,i} \equiv s^t_{Trg,i} / \sum_{i=1}^N s^t_{Trg,i}$ and rewrite equation (6.1) as follows:

$$\begin{aligned}
\ln f^1_{Trg} - \ln f^0_{Trg} &= \frac{1}{2} \sum_{i=1}^N \left(s^{*0}_{Trg,i} + s^{*1}_{Trg,i} \right) \left(\ln q^1_i - \ln q^0_i \right) \\
&\quad + \frac{1}{2} \sum_{i=1}^N [s^{*0}_{Trg,i}(\xi - 1) + s^{*1}_{Trg,i}(\xi - 1)] \left(\ln q^1_i - \ln q^0_i \right) \\
&\quad + \textit{Parameter-change component} \qquad (6.3)
\end{aligned}$$

where $\xi \equiv \sum_{i=1}^N q^t_i f^t_{Trg,i} / f^t_{Trg}$, which represents the degree of returns to scale. The first additive term in the right-hand side of (6.3) is the contribution of the change in q to the functional value difference, whereas the second additive term is the contribution of the scale effects.

Taking the antilogarithms of (6.3) yields:

$$\begin{aligned}
\frac{f^1_{Trg}}{f^0_{Trg}} &= \exp[\ln \frac{f^1_{Trg}}{f^0_{Trg}}] = \exp\left[\ln f^1_{Trg} - \ln f^0_{Trg}\right] \\
&= \left\{ \exp[\frac{1}{2} \sum_{i=1}^N \left(s^{*0}_{Trg,i} + s^{*1}_{Trg,i} \right) \left(\ln q^1_i - \ln q^0_i \right)] \right\} \\
&\quad \cdot \left\{ \exp[\frac{1}{2} \sum_{i=1}^N [s^{*0}_{Trg,i}(\xi - 1) + s^{*1}_{Trg,i}(\xi - 1)] \left(\ln q^1_i - \ln q^0_i \right)] \right\} \\
&\quad \cdot \exp(\textit{Parameter-change component}) \\
&= \prod_{i=1}^N \left(\frac{q^1_i}{q^0_i} \right)^{\frac{1}{2}\left(s^{*0}_{Trg,i} + s^{*1}_{Trg,i} \right)} \cdot \prod_{i=1}^N \left(\frac{q^1_i}{q^0_i} \right)^{\frac{1}{2}[s^{*0}_{Trg,i}(\xi-1) + s^{*1}_{Trg,i}(\xi-1)]} \\
&\quad \cdot \exp(\textit{Parameter-change component}) \qquad (6.4)
\end{aligned}$$

where $\prod_{i=1}^N \left(\frac{q^1_i}{q^0_i} \right)^{\frac{1}{2}\left(s^{*0}_{Trg,i} + s^{*1}_{Trg,i} \right)}$ is the *Törnqvist index number* of q. The Törnqvist index number is said to be "exact" for the translog function because it gives the same result as the ratio of two translog functions (net of the scale factor and parameter-change component). This index number belongs to the class of the so-called "superlative" index numbers, which are "exact" for a quadratic function (the "translog"), which can provide a second-order differential approximation to an arbitrary function.

This result does not require separability and homogeneity restrictions. It is commonly believed that only the case of the translog-based indicators of relative change does not rely on this restrictions (see, for example, Diewert, 2004, p. 450). This should be contrasted with the following results, where other indicators are constructed without imposing these restrictions.

THEOREM 6.2. *If a quadratic Box-Cox function with variable parameters is defined by (5.4) with $\rho \neq 0$ and $\lambda \neq 0$, then*

$$
\begin{aligned}
(f^1_{Q^{\rho,\lambda}})^\rho - (f^0_{Q^{\rho,\lambda}})^\rho = {} & \frac{1}{2}\frac{\rho}{\lambda}\sum_{i=1}^{N}\left\{ s^0_{Q^{\rho,\lambda}i}\frac{(f^0_{Q^{\rho,\lambda}})^\rho}{(q^0_i)^\lambda} + s^1_{Q^{\rho,\lambda}i}\frac{(f^1_{Q^{\rho,\lambda}})^\rho}{(q^1_i)^\lambda}\right\} \\
& \cdot[(q^1_i)^\lambda - (q^0_i)^\lambda] \\
& + \textit{Parameter change component} \qquad (6.5)
\end{aligned}
$$

where

$$
s^t_{Q^{\rho,\lambda}i} \equiv \frac{q^t_i f^t_{Q^{\rho,\lambda}i}}{f^t_{Q^\rho}} \quad \textit{with } t = 0, 1 \text{ and } f^t_{Q^{\rho,\lambda}i} \equiv \frac{\partial f^t_{Q^{\rho,\lambda}}}{q^t_i} \qquad (6.6)
$$

We note that, in general, $s^t_{Q^{\rho,\lambda}i} \neq s^t_i$, where $s^t_i \equiv q^t_i f^t_i / f^t$ and $f^t_i \equiv \partial f^t / \partial q^t_i$, except at the point of approximation. Moreover, $\sum_{i=1}^N s^t_{Q^{\rho,\lambda},i} = 1$ if $f^0_{Q^{\rho,\lambda}}$ and $f^1_{Q^{\rho,\lambda}}$ are homogeneous of degree one in q, since in this case $f^t_{Q^{\rho,\lambda}} = \sum_{i=1}^N q^t_i f^t_{Q^{\rho,\lambda},i}$ by Euler's theorem. If $\sum_{i=1}^N s^t_{Q^{\rho,\lambda},i} \neq 1$, then we can define $s^{*t}_{Q^{\rho,\lambda},i} \equiv s^t_{Q^{\rho,\lambda},i} / \sum_{i=1}^N s^t_{Q^{\rho,\lambda},i}$ and rewrite equation (6.5) as follows:

$$
\begin{aligned}
f^1_{Q^{\rho,\lambda}})^\rho - (f^0_{Q^{\rho,\lambda}})^\rho = {} & \frac{1}{2}\frac{\rho}{\lambda}\sum_{i=1}^{N}\left\{ s^{*0}_{Q^{\rho,\lambda}i}\frac{(f^0_{Q^{\rho,\lambda}})^\rho}{(q^0_i)^\lambda} + s^{*1}_{Q^{\rho,\lambda}i}\frac{(f^1_{Q^{\rho,\lambda}})^\rho}{(q^1_i)^\lambda}\right\} \\
& \cdot[(q^1_i)^\lambda - (q^0_i)^\lambda] \\
& + \frac{1}{2}\frac{\rho}{\lambda}\sum_{i=1}^{N}\left\{ s^{*0}_{Q^{\rho,\lambda}i}(\xi - 1)\frac{(f^0_{Q^{\rho,\lambda}})^\rho}{(q^0_i)^\lambda} + s^{*1}_{Q^{\rho,\lambda}i}(\xi - 1)\frac{(f^1_{Q^{\rho,\lambda}})^\rho}{(q^1_i)^\lambda}\right\} \\
& \cdot[(q^1_i)^\lambda - (q^0_i)^\lambda] \\
& + \textit{Parameter change component} \qquad (6.7)
\end{aligned}
$$

where $\xi \equiv \sum_{i=1}^N q^t_i f^t_{Q^{\rho,\lambda},i} / f^t_{Q^{\rho,\lambda}}$, which represents, by Euler's theorem, the degree of returns to scale. The first additive term in the right-hand side of

(6.7) is the contribution of the change in q to the functional value difference, whereas the second additive term is the contribution of the scale effects.

Dividing through (6.5) by $\left(f^0_{Q^{\rho,\lambda}}\right)^\rho$ yields

$$
\begin{aligned}
\frac{(f^1_{Q^{\rho,\lambda}})^\rho - (f^0_{Q^{\rho,\lambda}})^\rho}{(f^0_{Q^{\rho,\lambda}})^\rho} &= \frac{1}{2}\frac{\rho}{\lambda}\sum_{i=1}^{N}\left\{s^0_{Q^{\rho,\lambda}i}\frac{1}{(q_i^0)^\lambda} + s^1_{Q^{\rho,\lambda}i}\frac{(f^1_{Q^\rho})^\rho}{(f^0_{Q^\rho})^\rho}\frac{1}{(q_i^1)^\lambda}\right\}\frac{(q_i^1)^\lambda - (q_i^0)^\lambda}{(q_i^0)^\lambda}(q_i^0)^\lambda \\
&\quad + \textit{Parameter change component}/(f^0)^r \\
&= \frac{1}{2}\frac{\rho}{\lambda}\sum_{i=1}^{N}\left\{s^0_{Q^{\rho,\lambda}i} + s^1_{Q^{\rho}i}\frac{(f^1_{Q^{\rho,\lambda}})^\rho}{(f^0_{Q^{\rho,\lambda}})^\rho}\frac{(q_i^0)^\lambda}{(q_i^1)^\lambda}\right\}\frac{(q_i^1)^\lambda - (q_i^0)^\lambda}{(q_i^0)^\lambda} \\
&\quad + \textit{Parameter change component}/(f^0_{Q^{\rho,\lambda}})^\rho \qquad (6.8)
\end{aligned}
$$

Linear homogeneity and separability restrictions are not imposed, so that (6.5), (6.7), and (6.8) hold under very general conditions on the parameter values of $f^t_{Q^{\rho,\lambda}}$. We leave to the reader a further decomposition of (6.8) similar to (6.7). Let us consider, in particular, the following two special cases.

First case: General quadratic linear indicators, which are exact for a general quadratic linear function given by (5.12) with $\lambda = \rho/2$ and $\rho = 1$.

Using the decomposition

$$
(q_i^1)^{\frac{1}{2}} - (q_i^0)^{\frac{1}{2}} = (q_i^1 - q_i^0)\cdot\frac{1}{(q_i^0)^{\frac{1}{2}} + (q_i^1)^{\frac{1}{2}}}, \qquad (6.9)
$$

from (6.5), by defining $f^t_{Q^1} \equiv f^t_{Q^{\rho,\lambda}}$ with $\lambda = \rho/2$ and $\rho = 1$, we obtain

$$
\begin{aligned}
f^1_{Q^1} - f^0_{Q^1} &= \sum_{i=1}^{N}\left\{\frac{(q_i^0)^{\frac{1}{2}} f^0_{Q^1}}{[(q_i^0)^{\frac{1}{2}} + (q_i^1)^{\frac{1}{2}}]} + \frac{(q_i^1)^{\frac{1}{2}} f^1_{Q^1}}{[(q_i^0)^{\frac{1}{2}} + (q_i^1)^{\frac{1}{2}}]}\right\}(q_i^1 - q_i^0) \\
&\quad + \textit{Parameter-change component} \qquad (6.10)
\end{aligned}
$$

Dividing through (6.10) by $f^0_{Q^1}$ and replacing $(q^1_i - q^0_i)$ with $[(q^1_i - q^0_i)/q^0_i] \cdot q^0_i$ yields

$$\begin{aligned} \frac{f^1_{Q^1} - f^0_{Q^1}}{f^0_{Q^1}} &= \sum_{i=1}^{N} \left\{ s^0_{Q^1 i} \frac{(q^0_i)^{\frac{1}{2}}}{(q^0_i)^{\frac{1}{2}} + (q^1_i)^{\frac{1}{2}}} + s^1_{Q^1 i} \frac{f^1_{Q^1}}{f^0_{Q^1}} \frac{q^0_i}{(q^1_i)^{\frac{1}{2}}[(q^0_i)^{\frac{1}{2}} + (q^1_i)^{\frac{1}{2}}]} \right\} \frac{q^1_i - q^0_i}{q^0_i} \\ &\quad + \textit{Parameter-change component}/f^0_{Q^1} \qquad (6.11) \end{aligned}$$

(where $s^t_{Q^1 i} \equiv q^t_i f^t_{Q^1 i}/f^t_{Q^1}$ with $f^t_{1i} \equiv \partial f^t_{Q^1}/\partial q^t_i$)

We can call *general quadratic linear indicators* of absolute and relative functional value differences the indicators given, respectively, by the right-hand side of (6.10) and (6.11), which are exact for a general quadratic linear function corresponding to (5.12), where $\rho = 1$ and $\lambda = 1/2$, and do not rely on homogeneity and separability restrictions. This is a rather useful result, since it widens the applicability of this type of indicators. This can be contrasted with the same case where $r = 1$ and $\lambda = 1/2$ that was examined under linear homogeneity and separability restrictions by Diewert (2002, pp. 77-80), who showed that the counterpart index number, expressed in ratio terms is the implicit Walsh index number, which in turn is exact for a homogeneous quadratic linear function.

Second case: General Konüs-Byushgens indicator, which is exact for a general Konüs-Byushgens function given by (5.12) with $\lambda = \rho/2$ and $\rho = 2$.

Let us define $f^1_{Q^2} \equiv f^t_{Q^{\rho,\lambda}}$ with $\lambda = \rho/2$ and $\rho = 2$. Using the general decomposition

$$[f^1_{Q^2}]^2 - [f^0_{Q^2}]^2 = (f^1_{Q^2} - f^0_{Q^2}) \cdot (f^1_{Q^2} + f^0_{Q^2}), \qquad (6.12)$$

by dividing through (6.5) by $(f^1_{Q^2} + f^0_{Q^2})$ and rearranging terms, we obtain

$$\begin{aligned} f^1_{Q^2} - f^0_{Q^2} &= \sum_{i=1}^{N} \left(\frac{f^0_{Q^2 i} \cdot f^0_{Q^2}}{f^1_{Q^2} + f^0_{Q^2}} + \frac{f^1_{Q^2 i} \cdot f^1_{Q^2}}{f^1_{Q^2} + f^0_{Q^2}} \right)(q^1_i - q^0_i) \\ &\quad + \textit{Parameter-change component}/(f^1_{Q^2} + f^0_{Q^2}) \quad (6.13) \end{aligned}$$

where $f^t_{Q^2 i} \equiv \partial f^t_{Q^2}/\partial q^t_i$ for $t = 0, 1$.

Dividing through (6.13) by $f^0_{Q^2}$ and replacing $(q^1_i - q^0_i)$ with $[(q^1_i - q^0_i)/q^0_i] \cdot q^0_i$ yield

$$\begin{aligned} \frac{f^1_{Q^2} - f^0_{Q^2}}{f^0_{Q^2}} &= \sum_{i=1}^{N} \left\{ s^0_{Q^2 i} \frac{f^0_{Q^2}}{f^1_{Q^2} + f^0_{Q^2}} + s^1_{Q^2 i} \frac{q^0_i}{q^1_i} \frac{(f^1_{Q^2})^2}{f^0_{Q^2}(f^1_{Q^2} + f^0_{Q^2})} \right\} \frac{q^1_i - q^0_i}{q^0_i} \\ & +\textit{Parameter-change component}/(f^1_{Q^2} + f^0_{Q^2}) \qquad (6.14) \end{aligned}$$

$$\text{where } s^t_{Q^2 i} \equiv q^t_i f^t_{2i}/f^t_{Q^2} \quad \text{with } f^t_{2i} \equiv \partial f^t_{Q^2}/\partial q^t_i.$$

We can call *general Konüs-Byushgens indicators* of absolute and relative functional value differences the indicators given, respectively, by the right-hand side of (6.13) and (6.14), which are exact for a general Konüs-Byushgens quadratic function corresponding to (5.12) where $\rho = 2$ and $\lambda = 1$ and do not rely on homogeneity and separability restrictions. Also this result is useful to widen the applicability of this type of indicators. This can be contrasted with the same case where $\rho = 2$ and $\lambda = 1$, that was examined under linear homogeneity and separability restrictions by Reinsdorf, Diewert, and Ehemann (2000, pp. 4-6) and Diewert (2002, pp. 72-76), who showed that the counterpart index number (expressed in ratio terms) is the Fisher "ideal" index number, which in turn is exact for a homogeneous Konüs-Byushgens quadratic function.

THEOREM 6.3. *If the homothetic (homogeneous of degree r/ρ) function $f_{Q^{\rho,\lambda}}$ is defined by (5.18)-(5.19), where $\rho \neq 0$ and $\lambda \neq 0$, then, for all q^0_i and q^1_i,*

$$\frac{f^1_{Q^{\rho,\lambda}}}{f^0_{Q^{\rho,\lambda}}} = I_q \cdot I_Y \cdot I_T \qquad (6.15)$$

where, setting $r = 2\lambda$ and $\sigma^t = (1 + \rho a^t_0)^{\frac{1}{\rho}}$ for $t = 0, 1$,

$$I_q \equiv \left[\frac{\sum_{i=1}^{N} s^0_{Q^r i} \frac{(q^1_i)^{r/2}}{(q^0_i)^{r/2}}}{\sum_{i=1}^{N} s^1_{Q^r i} \frac{(q^1_i)^{-r/2}}{(q^0_i)^{-r/2}}} \right]^{\frac{1}{r}} = \left[\frac{\sum_{i=1}^{N} s^0_{Q^{\rho,\lambda} i} \frac{(q^1_i)^{r/2}}{(q^0_i)^{r/2}}}{\sum_{i=1}^{N} s^1_{Q^{\rho,\lambda} i} \frac{(q^1_i)^{-r/2}}{(q^0_i)^{-r/2}}} \right]^{\frac{1}{r}} \qquad (6.16)$$

$$I_Y \equiv \left[\frac{\sum_{i=1}^{N} s^0_{Q^{\rho,\lambda}i} \frac{(q^1_i)^{r/2}}{(q^0_i)^{r/2}}}{\sum_{i=1}^{N} s^1_{Q^{\rho,\lambda}i} \frac{(q^1_i)^{-r/2}}{(q^0_i)^{-r/2}}} \right]^{\frac{1}{\rho}-\frac{1}{r}} \tag{6.17}$$

$$I_T \equiv \frac{\sigma^1}{\sigma^0} \tag{6.18}$$

with

$$s^t_{Q^r i} \equiv q^t_i \cdot \frac{\partial f_{Q^r}}{\partial q^t_i} / f_{Q^r} \tag{6.19}$$

$$= s^t_{Q^{\rho,\lambda}i} \equiv q^t_i \cdot \frac{(\partial f_{Q^{\rho,\lambda}} / \partial q^t_i)}{\sum_{i=1}^{N} q^t_i \cdot (\partial f_{Q^{\rho,\lambda}} / \partial q^t_i)} \tag{6.20}$$

The index I_q defined by (6.16) is a *quadratic mean-of-order-r type index number* of q, which is "exact" for the quadratic mean-of-order-r aggregator function (5.19) because it is identically equal to the ratio between the two functional values of this function at $t = 0$ and $t = 1$, whereas I_Y defined by (6.17) is the index number of scale effects, and I_T defined by (6.18) is the index number of the effects of parameter changes. Being "exact" for a function that can provide a second-order differential approximation to an arbitrary function, the index I_q belongs to the class of the so-called "superlative" index numbers.

Note that the quadratic mean-of-order-r index number defined by (6.16) can be contrasted with the following quadratic mean-of-order-r index number defined by Diewert (1976, pp. 130-131):

$$I^D_q \equiv \left[\frac{\sum_{i=1}^{N} s^0_i \frac{(q^1_i)^{\frac{r}{2}}}{(q^0_i)^{\frac{r}{2}}}}{\sum_{i=1}^{N} s^1_i \frac{(q^1_i)^{-\frac{r}{2}}}{(q^0_i)^{-\frac{r}{2}}}} \right]^{\frac{1}{r}} \tag{6.21}$$

where s^0 and s^1 are actually "observed". By defining these weights as value shares, equation (6.21) reduces to well-known index numbers with particular values of r. Diewert (1976, p. 135) noted that, if $r = 1$, then (6.21) reduces to the implicit Walsh (1901, p. 105) index number, and, if $r = 2$, then (6.21) reduces to the Fisher (1922) "ideal" index number.

The indexes I_q and I_q^D differ in the weights used. Even if these two weight systems are numerically equal at the point of approximation, say at $t = 0$, so that $s^0_{Q^r} = s^0$, they may be substantially different at other points under comparison. This is sufficient to make the two index numbers (6.16) and (6.21) rather different. In order to ensure that $s^0_{Q^r}$ must be equal to s^0 at all values of r, the parameters of the underlying aggregator function should adjust to the changes in r. Consequently, $s^1_{Q^r}$ is also a function of r. This means that, using the same s^1 with different values of r contradicts the assumption of second-order approximation that is supposed to be provided by I_q^D. It is, therefore, I_q, rather than I_q^D, that must be called "superlative" according to the meaning assigned by Diewert (1976, p. 117) to this term. This will be shown more analytically and illustrated by a numerical example in the remainder of this section.

The function $f_{Q^r}(q)$ defined by (5.19) provides a second-order differential approximation to an arbitrary function $f(q)$ around q^0 if

$$f(q^0) = f_{Q^r}(q^0) \tag{6.22}$$

$$\nabla_z f(q^0) = \nabla_z f_{Q^r}(q^0) \tag{6.23}$$

$$\nabla_z^2 f(q^0) = \nabla_z^2 f_{Q^r}(q^0) \tag{6.24}$$

The equations (6.22)-(6.24) are satisfied for certain values of the parameters of f_{Q^r} for a given r, as established by the following result:

THEOREM 6.4. *If the aggregator function $f_{Q^r}(q)$ defined by (5.19) provides a second-order approximation to the arbitrary aggregator function $f(q)$ around q^*, then its parameters must be set equal to*:

$$\alpha_{ij} = \frac{f^*_{ij} - \frac{1-r}{f^*} f^*_i f^*_j}{\frac{r}{2}\left[f^*\right]^{1-r} (q^*_i q^*_j)^{\frac{r}{2}-1}} \qquad 1 \le i < j \le N \tag{6.25}$$

$$\alpha_{ii} = \frac{f^*_i - (q^*_i)^{\frac{r}{2}-1} \sum_{j \neq i}^{N} [f_{Q^r}(q^*)]^{(1-r)} \alpha^*_{ij} (q^*_j)^{\frac{r}{2}-1}}{[f_{Q^r}(q^*)]^{(1-r)} (q^*_i)^{r-2}} \qquad 1 \le i \le N \tag{6.26}$$

From the above result it is evident that the parameters of a second-order approximating quadratic function $f_{Q^r}(q)$ are univocally determined as functions of r.

Furthermore, note that in general,

$$f(q^1) \neq f_{Q^r}(q^1) \tag{6.27}$$
$$\nabla f(q^1) \neq \nabla f_{Q^r}(q^1) \tag{6.28}$$
$$\nabla^2 f(q^1) \neq \nabla^2 f_{Q^r}(q^1) \tag{6.29}$$

Therefore, given the definition (6.19), in general

$$s^1 \neq s^1_{Q^r} \tag{6.30}$$

where $s^1_i \equiv q^1_i \frac{\partial f(q)}{\partial q^1_i} / f(q)$.

When r changes, say from r^* to r^{**}, then in general

$$\nabla f_{Q^{r^*}}(z^1) \neq \nabla f_{Q^{r^{**}}}(z^1) \tag{6.31}$$

and, consequently,

$$s^1 \neq s^1_{Q^{r^*}} \neq s^1_{Q^{r^{**}}} \tag{6.32}$$

For each value $r = r^*$, there is a set of particular parameters $\alpha^*_{ij} = \alpha^*_{ij}(r^*)$ such that the function defined by

$$f_{Q^{r^*}}(q) \equiv [\Sigma^N_{i=1}\Sigma^N_{j=1}\alpha^*_{ij} q_i^{r/2} q_j^{r/2}]^{1/r} \tag{6.33}$$

satisfies (6.22)-(6.24). It is immediate to note that all the functions $f_{Q^{r^*}}(q)$ defined over the domain of r^* approximate each other up to the second order. Consequently, the corresponding "exact" index numbers defined by (6.16), where $r = r^*$, also approximate each other up to the second order. This is ensured by the adjustment of the shares $s^t_{Q^r}$ to the value of r. If, instead, the same observed shares s^t are used for different values or r, as in (6.21), then hybrid formulae are used which cannot be interpreted as superlative index numbers providing second-order differential approximations to an arbitrary function and to each other. The following example is a numerical representation of the differences between the two types of index numbers.

EXAMPLE 6.1.

Let us consider the case of three elements, $N = 3$, with $q_i^0 = 1.0$ for $i = 1, 2, 3$ so that , in the case of a linearly homogeneous function,

$$f(q^0) = 1.0 \tag{6.34}$$

and $q_1^1 = 2.0$, $q_2^1 = 1.7$, and $q_3^1 = 1.05$. Moreover, let assume that, at the observation point $t = 0$, the first derivatives of the unknown function $f(q^0)$ are the following:

$$f_1 \equiv \frac{\partial f}{\partial q_1} = 0.53 \tag{6.35}$$

$$f_2 \equiv \frac{\partial f}{\partial q_2} = 0.37 \tag{6.36}$$

$$f_3 \equiv \frac{\partial f}{\partial q_3} = 0.10 \tag{6.37}$$

At the same observation point, let us assume that the Hessian matrix of the function $f(q^0)$ is the following:

$$\nabla^2 f(q^0) = \begin{bmatrix} 0.0376 & -0.0088 & -0.0288 \\ -0.0088 & 0.0344 & -0.0256 \\ -0.0288 & -0.0256 & 0.0544 \end{bmatrix} \tag{6.38}$$

Therefore, at the point $t = 0$ and $t = 1$, the weights s_i^t are

$$s_1^0 \equiv (q_1^0 \cdot f_1)/f = 0.53 \qquad s_1^1 \equiv (q_1^1 \cdot f_1)/f = 0.69 \tag{6.39}$$

$$s_2^0 \equiv (q_2^0 \cdot f_2)/f = 0.37 \qquad s_2^1 \equiv (q_2^1 \cdot f_2)/f = 0.30 \tag{6.40}$$

$$s_3^0 \equiv (q_3^0 \cdot f_3)/f = 0.10 \qquad s_3^1 \equiv (q_3^1 \cdot f_3)/f = 0.01 \tag{6.41}$$

Let us now assume that a quadratic mean-of-order-r aggregator function of the type defined by (5.19) has such parameters α_{ij}^* for a given r^*

Table 2 – Parameters, weights, and numerical values of the quadratic mean-of-order *r* function

	$r = 1$	$r = 2$	$r = 4$	$r = 10$	$r = 20$	$r = 40$	$r = 100$	$r = 1000$	$r = ?$
α_{11}	0.6052	0.3185	0.1751	0.0891	0.0605	0.0461	0.0375	0.0324	0.0318
α_{22}	0.4388	0.1713	0.0376	-0.0427	-0.0694	-0.0828	-0.0908	-0.0957	-0.0962
α_{33}	0.2088	0.0644	-0.0078	-0.0511	-0.0656	-0.0728	-0.0771	-0.0797	-0.0800
$\alpha_{12} = \alpha_{21}$	-0.0176	0.1873	0.2897	0.3512	0.3717	0.3820	0.3881	0.3918	0.3922
$\alpha_{13} = \alpha_{31}$	-0.0576	0.0242	0.0651	0.0896	0.0978	0.1019	0.1044	0.1058	0.1060
$\alpha_{23} = \alpha_{32}$	-0.0512	0.0114	0.0427	0.0615	0.0677	0.0709	0.0727	0.0739	0.0740
$s^1_{Q^r,1}$	0.6057	0.6025	0.6045	0.6188	0.6544	0.8058	0.9970	1.0000	1.0000
$s^1_{Q^r,2}$	0.3570	0.3539	0.3567	0.3698	0.3448	0.1942	0.0030	0.0000	0.0000
$s^1_{Q^r,3}$	0.0373	0.0437	0.0388	0.0114	0.0008	0.0000	0.0000	0.0000	0.0000
Q_r	1.8069	1.8044	1.8066	1.8256	1.8474	1.8750	1.9355	1.9932	2.0000
Q^D_r	1.8336	1.8381	1.8440	1.8389	1.7522	1.6013	1.5078	1.4549	1.44914

Figure 4:

that it provides a second-order differential approximation to f at q^0. Table 6.1 tabulates the values of these parameters in correspondence of certain values of r, so that the quadratic mean-of-order-r aggregator functions f_{Q^r} approximate f up to the second order. The shares $s^0_{Q^r}$ and $s^1_{Q^r}$ are also calculated. The corresponding "exact" and "superlative" index numbers I_r are compared with the hybrid index numbers I^D_r, which are constructed using the same "observed" shares for different values of r. From the numerical example, it is evident that the two index numbers diverge increasingly from one another as r increases. The "really" superlative index numbers tend to increase as r increases after a certain turning point, whereas the "hybrid" index numbers tend to decrease as r increases after another turning point. Both the "really" and "hybrid" superlative index numbers have asymptotic values as $r \to \infty$. In the case considered in this numerical example, the former tends to a value equal to 2.0, which is higher than that obtained at $r = 1$ by more than 10 per cent, whereas the latter tends to a value equal to 1.44914, which is lower than that obtained at $r = 1$ by more than 20 per cent (see Table 2).

These results show that the theory of exact and superlative index numbers has to be widely reconsidered and shed a new light to the apparently paradoxical findings recently obtained by Hill (2002, 2005).

8. Conclusion

This paper has reached new and more general results in the field of exact and superlative index numbers. It has widen the scope of applicability of these index numbers by showing that they can be justified under weaker assumptions than those traditionally made. However, it is also shown that these approximating index numbers are rarely applicable in practice. In fact, a superlative index number is defined by using weights derived from the function for which it is "exact". If, instead, the weights are derived from observed data, which in turn are assumed to be consistent with the "true" unknown arbitrary function, then the resulting index number is a hybrid formula that may turn out to be very different from a second-order approximating index number.

The decomposition procedures described in this paper require that the weights used to aggregate absolute or relative changes in the observed variables be known or "approximated" in some way. In particular, with the quadratic mean-of-order-r index numbers, for the given value of r, knowledge of the weights is essential for the computation of index numbers. If the corresponding function for which these index numbers or indicators are "exact" is *not* the "true" function, then the required weights remain unknown. It is remarkable that these may be far from being approximated by the weights derived from the observed data. Therefore, these index numbers vary widely from the "true" value and from each other if wide changes in r are not accompanied by the necessary parameter adjustments. This has consequences that are particularly relevant in common practice where observed or estimated weights are used. These index numbers turn out to be hybrid and far from being really "superlative" in Diewert's (1976) sense.

We conclude that, since using the "observed" data normally available it is not possible to rely on the second-order differential approximation paradigm, it would be more appropriate to construct a range of alternative index numbers (including those that are not superlative), which are to be considered all equally valid candidates as good approximation to the true unknown index number, rather than follow the traditional search for only *one* optimal formula. In intertemporal comparisons dealing with time series data, statistical agencies might further reduce in some degree the spread between alternative index numbers by adopting, for each of them, the *chaining principle* and changing the base period more frequently. However, they have to take into account that chained indexes do not satisfy, in general, certain required properties. In intertemporal comparisons dealing with decennial census data and interspatial comparisons the reduction of the spread is more problematic.

Appendix A

Observable economic variables

The propositions presented thus far are theorems in numerical analysis rather than economics. The numerical values of the variables involved are assumed to be either known or derivable from some sources of information. In economics, the contexts where index numbers and indicators are usually applied are those regarding production and consumption activities. The index numbers and indicators are therefore defined with reference to production or transformation and utility functions and their dual counterparts represented by *value functions* such as cost, revenue and profit functions. All these functions are characterized by certain properties and the variables concerned are not always "observable". The following cases are examined:

(i) *The case of the production function*

If the function $f^t(x)$ is homogeneous of degree 1 or less than 1 and represent a transformation or production function characterized by the usual regularity properties so that

$$y \quad = \quad f^t(x) \tag{A.1}$$

where y is a scalar measure of the output quantity, and x is a vector of input quantities, then the first-order conditions for profit maximization imply (dropping the superscript t to simplify notation)

$$\frac{\partial f}{\partial x_i} \quad = \quad \frac{w_i}{p} \quad \text{for all } i\text{'s} \tag{A.2}$$

where p is a scalar measure of the output price. Therefore,

$$s_i \quad \equiv \quad \frac{\partial f}{\partial x_i}\frac{x_i}{y} = \frac{w_i \cdot x_i}{p \cdot y} \tag{A.3}$$

Under constant returns to scale and perfect competition, $p \cdot y = \sum_{i=1}^{N} w_i \cdot x_i$, so that $s_i = w_i \cdot x_i / \sum_{i=1}^{N} w_i \cdot x_i$ and $\sum_{i=1}^{N} s_i = 1$.

In the general case where $p \cdot y \gtreqless \sum_{i=1}^{N} w_i \cdot x_i$, if we want to construct separately index numbers or indicators of differences in quantities and returns to scale, then it might be useful to decompose the foregoing weights as follows:

$$s_i = \frac{w_i \cdot x_i}{\sum_{i=1}^{N} w_i \cdot x_i} + \frac{w_i \cdot x_i}{\sum_{i=1}^{N} w_i \cdot x_i} \cdot (\xi - 1) \tag{A.4}$$

where $\xi \equiv \sum_{i=1}^{N} w_i \cdot x_i)/p \cdot y$ represents the degree of the returns to scale. It is obtained using the additional information on output prices and quantities. The second term of the right-hand side of equation (A.4) represents the weight capturing the effect of returns to scale (see Caves, Christensen, and Diewert, 1982, pp. 1405-1406 and p. 1408 for a similar decomposition[19]).

Since prices and quantities of inputs and outputs are usually observable, all the necessary information for the calculation of index numbers and related indicators is available.

(ii) The case of the utility function

Let the function $f^t(x)$ represent a utility function characterized by the usual regularity properties so that

$$u = f^t(x) \tag{A.5}$$

where u represents utility, and x is a vector of the quantities consumed. Utility is typically unobserved, so that at least this important variable is not available for the index number construction. However, if the consumer has a utility-maximizing behavior subject to a budget constraint, then the

[19] The variable parameter ξ (in our notation) is obtained with equation (47) in Caves *et. al.* (1982, p. 1406), where it is, however, misprinted: a simple mathematical derivation reveals that it is incorrectly inverted (the correct form is reported at page 1408 of the same article).

first-order conditions imply (dropping the superscript t to simplify notation)

$$\frac{\partial f}{\partial x_i} = \lambda w_i \quad \text{for all } i\text{'s} \tag{A.6}$$

We sum these conditions multiplied by quantities in order to obtain:

$$\sum_{i=1}^{N} \frac{\partial f}{\partial x_i} x_i = \lambda \sum_{i=1}^{N} w_i x_i \tag{A.7}$$

which can be solved for the Lagrange multiplier as follows:

$$\lambda = \frac{\sum_{i=1}^{N} \frac{\partial f}{\partial x_i} x_i}{\sum_{i=1}^{N} w_i x_i} \tag{A.8}$$

Substituting the Lagrange multiplier back into the first-order conditions yields:

$$\frac{\partial f}{\partial x_i} = \left(\frac{\sum_{i=1}^{N} \frac{\partial f}{\partial x_i} x_i}{\sum_{i=1}^{N} w_i x_i} \right) w_i \quad \text{for all } i\text{'s} \tag{A.9}$$

from which we derive the Hotelling (1935, p.71)-Wold (1944, pp. 69-71; 1953, p. 145) identity

$$\frac{\partial f}{\partial x_i} \frac{1}{\sum_{i=1}^{N} \frac{\partial f}{\partial x_i} x_i} = \frac{w_i}{\sum_{i=1}^{N} w_i x_i} \qquad \text{for all } i\text{'s} \tag{A.10}$$

If we assume linear homogeneity of the utility function (implying marginal utility of income equal to 1), then, by Euler's theorem, $\sum_{i=1}^{N} \frac{\partial f}{\partial x_i} x_i = f(x)$ and, consequently, the foregoing equation becomes

$$s_i \equiv \frac{\frac{\partial f}{\partial x_i} x_i}{f(x)} = \frac{w_i x_i}{\sum_{i=1}^{N} w_i x_i} \quad \text{for all } i\text{'s} \tag{A.11}$$

In this last case, the weights s_i can be calculated using observed data on prices and quantities.

(iii) The case of the value functions (cost, revenue, and profit functions)

Let a value function (a cost, or revenue, or profit function) be represented by a differentiable function $e^t(p)$, where p is a vector of prices. By Hicks (1946, p. 331)-Samuelson (1947, p. 68)-Shephard (1953, p. 11)-Hotelling's lemma (1932, p. 594), we obtain directly the optimal levels of quantities through differentiation (dropping the superscript t to simplify notation):

$$q_i = \frac{\partial e}{\partial p_i} \tag{A.12}$$

where q_i is the i^{th} element of an N-dimensional vector of quantities.

Irrespective of the technology of production or utility function, the value function is always linearly homogeneous in prices. By Euler's theorem, $e(p) = \sum_{i=1}^{N} p_i q_i$. Therefore, by dividing both sides of the equation (A.12) by $e(p)$ we obtain

$$\frac{q_i}{\sum_{i=1}^{N} p_i q_i} = \frac{\frac{\partial e}{\partial p_i}}{e(p)} \quad \text{for all } i\text{'s} \tag{A.13}$$

Multiplying the foregoing equations by the respective prices p_i's yields

$$\frac{p_i q_i}{\sum_{i=1}^{N} p_i q_i} = \frac{p_i \frac{\partial e}{\partial p_i}}{e(p)} = s_i \quad \text{for all } i\text{'s} \tag{A.14}$$

Here, again, s_i and the numerical value of $e(p)$ can be calculated using observed data on prices and quantities.

Appendix B

Proofs of theorems

Proof of Lemma (2.1) (Accounting for Functional Value Differences).

Let us consider an arbitrary function $f(z)$ of one single variable. From the Taylor series expansion for f around z^0, the following equation can be obtained:

$$\begin{aligned} f(z^1) - f(z^0) &= f'(z^0)(z^1 - z^0) + \frac{1}{2!} f''(z^0)(z^1 - z^0)^2 + ... \\ &\quad + \frac{1}{n!} f^{(n)}(z^0)(z^1 - z^0)^n + R_n^0(z^0, z^1) \\ &= \sum_{m=1}^{n} \frac{1}{m!} f^{(m)}(z^0)(z^1 - z^0)^m + R_n^0(z^0, z^1) \qquad \text{(B.1)} \end{aligned}$$

where $R_n^0(z^0, z^1)$ is the remainder term. Similarly, from the Taylor series expansion for f around z^1, the following equation can be obtained:

$$\begin{aligned} f(z^0) - f(z^1) &= f'(z^1)(z^0 - z^1) + \frac{1}{2!} f''(z^1)(z^0 - z^1)^2 + ... \\ &\quad + \frac{1}{n!} f^{(n)}(z^1)(z^0 - z^1)^n + R_n^1(z^0, z^1) \\ &= \sum_{m=1}^{n} \frac{1}{m!} f^{(m)}(z^1)(z^0 - z^1)^m + R_n^1(z^0, z^1) \qquad \text{(B.2)} \end{aligned}$$

Multiplying through the foregoing equation by -1 and rearranging terms yield

$$f(z^1) - f(z^0) = -\sum_{m=1}^{n} (-1)^m \frac{1}{m!} f^{(m)}(z^1)(z^1 - z^0)^m - R_n^1(z^0, z^1) \qquad \text{(B.3)}$$

Using $(1 - \theta)$ and θ as weights, the weighted average of (B.1) and (B.3) is given by

$$\begin{aligned} f(z^1) - f(z^0) &= \sum_{m=1}^{n} \left[(1 - \theta) \frac{1}{m!} f^{(m)}(z^0) - (-1)^m \theta \frac{1}{m!} f^{(m)}(z^1) \right] (z^1 - z^0)^m \\ &\quad + (1 - \theta) R_n^0(z^0, z^1) - \theta R_n^1(z^0, z^1) \qquad \text{(B.4)} \end{aligned}$$

From (B.4), it follows that

$$
\begin{aligned}
f(z^1) - f(z^0) &= \left[(1-\theta)\, f'(z^0) + \theta f'(z^1)\right]^T (z^1 - z^0) \\
&\quad + (1-\theta)\ R_1^{(z^0)} - \theta\ R_1^{(z^1)} \qquad \text{(B.5)}
\end{aligned}
$$

Let us choose a value of θ, say θ^*, that minimizes the squared term $[(1-\theta)\, R_1^0(z^0, z^1) - \theta R_1^1(z^0, z^1)]^2$. Since this term is convex in θ, the necessary and sufficient condition for its minimization is that its first derivative with respect to θ vanishes, so that

$$
\begin{aligned}
& -2\left[(1-\theta)\, R_1^0(z^0, z^1) - \theta R_1^1(z^0, z^1)\right] \left[R_1^0(z^0, z^1) + R_1^1(z^0, z^1)\right] \\
= \ & 0 \qquad \text{(B.6)}
\end{aligned}
$$

from which, provided that $R_1^0(z^0, z^1) + R_1^1(z^0, z^1) \neq 0$,

$$
\theta = \theta^*(z^0, z^1) \equiv \frac{R_1^0(z^0, z^1)}{R_1^0(z^0, z^1) + R_1^1(z^0, z^1)} \qquad \text{(B.7)}
$$

The case of many variables follows in a similar manner using the directional derivatives. In particular,

$$
f(z^1) - f(z^0) = \left[(1-\theta)\, \nabla f(z^0) + \theta \nabla f(z^1)\right]^T (z^1 - z^0) \qquad \text{(B.8)}
$$

where, if $R_1^0(z^0, z^1) + R_1^1(z^0, z^1) \neq 0$, then $\theta = \theta^*$ (so that $(1-\theta^*)\, R_1^0(z^0, z^1)$ $-\ \theta^* R_1^1(z^0, z^1) = 0$), or, if $R_1^{(z^0)} = R_1^{(z^1)} = 0$, then θ may take any value as a number.

Proof of Corollary (2.1) (Accounting for Functional Value Ratios).

Applying the accounting procedure (2.1) to $\phi(z)$ and dividing through both sides by $\phi(z^0)$ yield

$$\frac{\phi(z^1)-\phi(z^0)}{\phi(z^0)} = \sum_{i=1}^{N}\left[(1-\theta)\frac{\partial\phi(z^0)}{\partial z_i}\cdot\frac{1}{\phi(z^0)}+\theta\frac{\partial\phi(z^1)}{\partial z_i}\cdot\frac{1}{\phi(z^0)}\right] \cdot\frac{z_i^1-z_i^0}{z_i^0}\cdot z_i^0 \tag{B.9}$$

hence

$$\frac{\phi(z^1)}{\phi(z^0)}-1 = (1-\theta)\sum_{i=1}^{N}s_i^0\frac{z_i^1}{z_i^0}-(1-\theta)\sum_{i=1}^{N}s_i^0 +\theta\,\frac{\phi(z^1)}{\phi(z^0)}\sum_{i=1}^{N}s_i^1-\theta\,\frac{\phi(z^1)}{\phi(z^0)}\sum_{i=1}^{N}s_i^1\frac{z_i^0}{z_i^1} \tag{B.10}$$

(where $s_i^t \equiv [\partial\phi(z^t)/\partial z_i]\cdot z_i^t/\phi(z^t) = \frac{\partial F^{-1}[f(z)]/\partial z_i}{F^{-1}[f(z)]}\cdot z_i = \frac{[\partial f(z^t)/\partial z_i]\cdot z_i^t}{\sum_{i=1}^{N}[\partial f(z^t)/\partial z_i]\cdot z_i^t}$, since $\phi(z^t)=F^{-1}[f(z)]$)

$$= (1-\theta)\sum_{i=1}^{N}s_i^0\frac{z_i^1}{z_i^0}-(1-\theta) +\theta\,\frac{\phi(z^1)}{\phi(z^0)}-\theta\,\frac{\phi(z^1)}{\phi(z^0)}\sum_{i=1}^{N}s_i^1\frac{z_i^0}{z_i^1} \tag{B.11}$$

Rearranging (B.11) and solving for $\phi(z^1)/\phi(z^0)$ yield

$$\frac{\phi(z^1)}{\phi(z^0)} = \frac{\theta+(1-\theta)\sum_{i=1}^{N}s_i^0\frac{z_i^1}{z_i^0}}{(1-\theta)+\theta\sum_{i=1}^{N}s_i^1\frac{z_i^0}{z_i^1}} \tag{B.12}$$

Proof of Corollary (2.2) (Diewert's, 1976, p. 117, Quadratic Identity).

Sufficiency: From the definition (2.4) of the quadratic function f_Q, the first difference of this function is obtained as follows:

$$f_Q(z^1) - f_Q(z^0) = a_0 + a^T\ z^1 + \frac{1}{2} z^{1T} A z^1 - a_0 - a^T\ z^0 - \frac{1}{2} z^{0T} A z^0 \qquad \text{(B.13)}$$

By adding $\frac{1}{2}z^{0T}Az^1$ and subtracting $\frac{1}{2}z^{1T}Az^0$ (recall that $z^{0T}Az^1 = z^{1T}Az^0$ since $A^T = A$) and substituting $\frac{1}{2}z^{0T}Az^1$ with $(z^{0T}Az^1 - \frac{1}{2}z^{0T}Az^1)$ and $\frac{1}{2}z^{0T}Az^0$ with $(z^{0T}Az^0 - \frac{1}{2}z^{0T}Az^0)$, equation (B.13) can be rearranged into the following

$$f_Q(z^1) - f_Q(z^0) = (a + z^0 A)^T (z^1 - z^0) + \frac{1}{2}(z^1 - z^0)^T A (z^1 - z^0) \qquad \text{(B.14)}$$

which can also be derived directly from the Taylor series expansion of $f_Q(z)$ around the point z^0, since

$$\nabla_z f_Q(z^0) = a + z^0 A \qquad \text{(B.15)}$$

$$\nabla_z^2 f_Q(z^0) = A \qquad \text{(B.16)}$$

Similarly, from

$$f_Q(z^0) - f_Q(z^1) = a_0 + a^T\ z^0 + \frac{1}{2} z^{0T} A z^0 - a_0 - a^T\ z^1 - \frac{1}{2} z^{1T} A z^1 \qquad \text{(B.17)}$$

by adding $\frac{1}{2}z^{1T}Az^0$ and subtracting $\frac{1}{2}z^{0T}Az^1$ (recall, again, that $z^{1T}Az^0 = z^{0T}Az^1$ since $A^T = A$) and substituting $\frac{1}{2}z^{1T}Az^0$ with $(z^{1T}Az^0 - \frac{1}{2}z^{1T}Az^0)$ and $\frac{1}{2}z^{1T}Az^1$ with $(z^{1T}Az^1 - \frac{1}{2}z^{1T}Az^1)$, equation (B.17) can be rearranged into the following

$$f_Q(z^0) - f_Q(z^1) = (a + z^1 A)^T (z^0 - z^1) + \frac{1}{2}(z^0 - z^1)^T A (z^0 - z^1) \qquad \text{(B.18)}$$

which can also be derived directly from the Taylor series expansion of $f_Q(z)$ around the point z^1, since

$$\nabla_z f_Q(z^1) = a + z^1 A \quad \text{(B.19)}$$
$$\nabla_z^2 f_Q(z^1) = A \quad \text{(B.20)}$$

Multiplying through (B.18) by -1 and rearranging terms yields

$$\begin{aligned} & f_Q(z^1) - f_Q(z^0) \\ = \;& (a + z^1 A)^T (z^1 - z^0) - \frac{1}{2}(z^1 - z^0)^T A (z^1 - z^0) \end{aligned} \quad \text{(B.21)}$$

The arithmetic average of (B.14) and (B.21) is given by

$$\begin{aligned} f_Q(z^1) - f_Q(z^0) &= \frac{1}{2}[(a + z^0 A) + (a + z^1 A)]^T (z^1 - z^0) \\ &= \frac{1}{2}[\nabla_z f_Q(z^0) \ + \nabla_z f_Q(z^1)]^T (z^1 - z^0) \end{aligned} \quad \text{(B.22)}$$

Necessity: let us start from the following equation,

$$f(z^1) = f(z^0) + \frac{1}{2}\left[\nabla_z f(z^0) \ + \nabla_z f(z^1)\right]^T (z^1 - z^0) \quad \text{(B.23)}$$

where $\nabla f(z^0) \neq \nabla f(z^1)$ if $z^0 \neq z^1$ when f is thrice differentiable. Consider the polynomial of degree two $f_Q(z) \equiv \bar{a}_0 + \bar{a}\ z \ + z\ \bar{A}\ z$ for all z. The function $f_Q(z)$ is assumed to be tangent to $f(z)$ in correspondence to z^0 so that $\nabla f_Q(z^0) = \nabla f(z^0)$ and $f_Q(z^0) = f(z^0)$. Let us assume also that the numerical values of parameters of $f_Q(z)$ are such that $f_Q(z^1) = f(z^1)$ is satisfied. From the sufficiency condition,

$$f_Q(z^1) = f_Q(z^0) + \frac{1}{2}\left[\nabla f_Q(z^0) + \nabla f_Q(z^1)\right]^T (z^1 - z^0) \quad \text{(B.24)}$$

Substituting $f(z^0)$ and $f(z^1)$ to $f_Q(z^0)$ and $f_Q(z^1)$, respectively, the foregoing equation becomes

$$f(z^1) \;=\; f(z^0) + \frac{1}{2}\left[\nabla f(z^0) \;+ \nabla f_Q(z^1)\right]^T (z^1 - z^0) \qquad \text{(B.25)}$$

Since both (B.23) and (B.25) must hold simultaneously at z^0 and z^1, then $\nabla f(z^1) = \nabla f_Q(z^1)$ ($\neq \nabla f(z^0) = \nabla f_Q(z^0)$). Moreover, $f(z^1) = f_Q(z^1)$ by construction, therefore $f_Q(z)$ is tangent to $f(z)$ in correspondence of z^1 as well as of z^0. Since this result is to be valid for all the admissible pairs z^0 and z^1, $f(z)$ must have continuously the same gradients of a polynomial of degree at all z^0 and z^1. Hence, $f(z)$ itself must have the same quadratic functional form as that of $f_Q(z)$.

Proof of Corollary (2.3) (Accounting for Functional Value Ratios of a Quadratic Homothetic Function)

The proof is given in section 6, Theorem (6.3), within a more general context.

Proof of Lemma (2.2) (General Quadratic Approximation Lemma).

From (2.14) we have

$$\begin{aligned} &\textit{Error of approximation}\\ =\;& \left[(1-\theta)[\nabla_z f(z^0) + \theta\nabla_z f(z^1)\right]^T (z^1 - z^0)\\ &-\frac{1}{2}[\nabla_z f(z^0) + \nabla_z f(z^1)]^T (z^1 - z^0)\\ =\;& \left(\frac{1}{2} - \theta\right)[\nabla_z f(z^0) - \nabla_z f(z^1)](z^1 - z^0) \qquad \text{(B.26)}\end{aligned}$$

The first-order approximating (linear) function that is tangent to $f(z)$ at z^t is given by $f_L^t(z) = f(z^t) + \nabla_z f(z^t)^T(z - z^t) = a^t + b^t z$ (with $a^t = f(x^t) - \nabla_z f$ $(x^t)\ x^t$ and $b^t = \nabla_z f\ (x^t)$). Therefore

$$f_L^t(z) - f_L^t(z^t) \;=\; \nabla_z f(z^t)^T(z - z^t) \qquad \text{(B.27)}$$

Using (B.27) with $t = 0, 1$ and $z = z^0, z^1$, equation (B.26) becomes

$$\begin{aligned} &\textit{Error of approximation}\\ =\;& \left(\frac{1}{2} - \theta\right)\left\{\left[f_L^0(z^1) - f_L^0(z^0)\right] - \left[f_L^1(z^1) - f_L^1(z^0)\right]\right\} \qquad \text{(B.28)}\end{aligned}$$

Proof of Lemma (3.1) (Accounting for Functional Value Differences when Parameters or Functional Forms Differ)

Let us two arbitrary functions of one single variable, $f^0(z)$ and $f^1(z)$. These functions may differ in parameter values or even in their functional forms. From the Taylor series expansion for $f^0(z^1)$ around z^0, the following equation may be obtained:

$$f^0(z^1) - f^0(z^0) = f^{0\prime}(z^0)(z^1 - z^0) + R_1^0 \tag{B.29}$$

where $R_1^0 = R_1^0(z^0, z^1)$ is the remainder term of the first-order approximation.

Adding and subtracting $f^1(z^1)$ and rearranging terms, the foregoing equation becomes:

$$f^1(z^1) - f^0(z^0) = f^{0\prime}(z^0)(z^1 - z^0) + \left[f^1(z^1) - f^0(z^1)\right] + R_1^0 \tag{B.30}$$

Similarly, from the Taylor series expansion for $f^1(z^0)$ around z^1, the following equation may be obtained:

$$f^1(z^0) - f^1(z^1) = f^{1\prime}(z^1)(z^0 - z^1) + R_1^1 \tag{B.31}$$

where $R_1^1 = R_1^1(z^0, z^1)$ is the remainder term of the first-order approximation. By multiplying both sides by -1 and rearranging terms, equation (B.31) becomes:

$$f^1(z^1) - f^1(z^0) = f^{1\prime}(z^1)(z^1 - z^0) - R_1^1 \tag{B.32}$$

Adding and subtracting $f^0(z^0)$ and rearranging terms, the foregoing equation becomes:

$$f^1(z^1) - f^0(z^0) = f^{1\prime}(z^0)(z^1 - z^0) + \left[f^1(z^0) - f^0(z^0)\right] - R_1^1 \tag{B.33}$$

Using $(1-\theta)$ and θ as weights, where θ may take any real number as a value, the weighted average of (B.30) and (B.33) is given by

$$\begin{aligned} f^1(z^1) - f^0(z^0) &= \left[(1-\theta)f^{0\prime}(z^0) + \theta f^{1\prime}(z^0)\right](z^1 - z^0) \\ &\quad + \left\{\theta\left[f^1(z^0) - f^0(z^0)\right] + (1-\theta)\left[f^1(z^1) - f^0(z^1)\right]\right\} \\ &\quad + \left[(1-\theta)R_1^0 - \theta R_1^1\right] \end{aligned} \tag{B.34}$$

If $\theta = \theta^*$, with $\theta^* = \theta^*(z^0, z^1) \equiv \frac{R_1^0}{R_1^0 + R_1^1}$, then

$$\begin{aligned} f^1(z^1) - f^0(z^0) &= \left[(1-\theta)f^{0\prime}(z^0) + \theta f^{1\prime}(z^0)\right](z^1 - z^0) \\ &\quad + \textit{"Technical" change component (TC)} \end{aligned} \tag{B.35}$$

where

$$TC \equiv \theta^*\left[f^1(z^0) - f^0(z^0)\right] + (1-\theta^*)\left[f^1(z^1) - f^0(z^1)\right] \tag{B.36}$$

The case of many variables follows in a similar manner using the directional derivatives, thus obtaining (3.5)-(3.6).

Proof of Corollary (3.1) (Accounting for Functional Value Ratios between Arbitrary Differentiable Functions).

The proof of Corollary (3.1) is analogous to that of Corollary (2.1).

Proof of Corollary (3.2) (Accounting for Numerical Value Differences of Two Quadratic Functions Differing in Parameters).

The proof of Corollary (3.2) follows the footsteps of the proof of the "sufficiency" part of Corollary (2.2). From the definition (3.11) of the quadratic function f_Q^t, it is straightforward to obtain:

$$\begin{aligned} f_Q^1(z^1) - f_Q^0(z^0) &= (a^0 + z^0 A^0)(z^1 - z^0) \\ &\quad + (a_0^1 - a_0^0) + (a^1 - a^0)z^1 \\ &\quad - z^0 A^0 z^1 + \frac{1}{2}(z^0 A^0 z^0 + z^1 A^1 z^1) \end{aligned} \tag{B.37}$$

since

$$f_Q^1(z^1) - f_Q^0(z^0) = \left[f_Q^0(z^1) - f_Q^0(z^0)\right] + \left[f_Q^1(z^1) - f_Q^0(z^1)\right] \tag{B.38}$$

where

$$f_Q^0(z^1) - f_Q^0(z^0) = (a^0 + z^0 A^0)(z^1 - z^0) + \frac{1}{2}(z^1 - z^0)A^0(z^1 - z^0) \quad \text{(B.39)}$$

using the Taylor series expansion for f^0 around z^0,

$$f_Q^1(z^1) - f_Q^0(z^1) = (a_0^1 - a_0^0) + (a^1 - a^0)^T z^1 + \frac{1}{2}z^1(A^1 - A^0)z^1 \quad \text{(B.40)}$$

and

$$\begin{aligned} & \frac{1}{2}(z^1 - z^0)A^0(z^1 - z^0) + \frac{1}{2}z^1(A^1 - A^0)z^1 \\ = & -z^0 A^0 z^1 + \frac{1}{2}(z^0 A^0 z^0 + z^1 A^1 z^1) \end{aligned} \quad \text{(B.41)}$$

Similarly,

$$\begin{aligned} f_Q^0(z^0) - f_Q^1(z^1) = & (a^1 + z^1 A^1)(z^0 - z^1) \\ & + (a_0^0 - a_0^1) + (a^0 - a^1)z^0 \\ & - z^0 A^1 z^1 + \frac{1}{2}(z^0 A^0 z^0 + z^1 A^1 z^1) \end{aligned} \quad \text{(B.42)}$$

Since

$$f_Q^0(z^0) - f_Q^1(z^1) = \left[f_Q^1(z^0) - f_Q^1(z^1)\right] + \left[f_Q^0(z^0) - f_Q^1(z^0)\right] \quad \text{(B.43)}$$

where

$$f_Q^1(z^0) - f_Q^1(z^1) = (a^1 + z^1 A^1)(z^0 - z^1) + \frac{1}{2}(z^0 - z^1)A^1(z^0 - z^1) \quad \text{(B.44)}$$

using the Taylor series expansion for f^1 around z^1

$$f_Q^0(z^0) - f_Q^1(z^0) = (a_0^0 - a_0^1) + (a^0 - a^1)z^0 + \frac{1}{2}z^0(A^0 - A^1)z^0 \quad \text{(B.45)}$$

and

$$\begin{aligned} & \frac{1}{2}(z^1 - z^0)A^0(z^1 - z^0) + \frac{1}{2}z^1(A^1 - A^0)z^1 \\ = & -z^0 A^0 z^1 + \frac{1}{2}(z^0 A^0 z^0 + z^1 A^1 z^1) \end{aligned} \quad \text{(B.46)}$$

Multiplying both sides of (B.42) by -1 and rearranging terms yield

$$\begin{aligned} f_Q^1(z^1) - f_Q^0(z^0) = & (a^1 + z^1 A^1)(z^1 - z^0) \\ & + (a_0^1 - a_0^0) + (a^1 - a^0)z^0 \\ & + z^0 A^1 z^1 - \frac{1}{2}(z^0 A^0 z^0 + z^1 A^1 z^1) \end{aligned} \quad \text{(B.47)}$$

The arithmetic average of (B.37) and (B.47) is therefore

$$
\begin{aligned}
f_Q^1(z^1) - f_Q^0(z^0) &= \frac{1}{2}\left[\nabla f_Q^0(z^0) + \nabla f_Q^1(z^1)\right](z^1 - z^0) \\
&\quad +(a_0^0 - a_0^1) + (a^0 - a^1)^T \frac{1}{2}(z^0 + z^1) \\
&\quad +\frac{1}{2} z^0 (A^1 - A^0) z^1 \qquad \text{(B.48)}
\end{aligned}
$$

since $\nabla f_Q^0(z^0) = a^0 + z^0 A^0$, $\nabla f_Q^1(z^1) = a^1 + z^1 A^1$.

Moreover, by definition,

$$a_0^0 - a_0^1 = (\alpha_0^1 + k^1\beta^1 + \frac{1}{2}k^1 B^1 k^1) - (\alpha_0^0 + k^0\beta^0 + \frac{1}{2}k^0 B^0 k^0) \qquad \text{(B.49)}$$

$$(a^0 - a^1)\frac{1}{2}(z^0 + z^1) = (\alpha^1 + k^1\Gamma^1 - \alpha^0 - k^0\Gamma^0)\frac{1}{2}(z^0 + z^1) \qquad \text{(B.50)}$$

The sum of (B.49) and (B.50) is

$$\left(a_0^0 - a_0^1\right) + (a^0 - a^1)\frac{1}{2}(z^0 + z^1) =$$

$$
\begin{aligned}
&(\alpha_0^0 + \alpha_0^1) + (\alpha^0 - \alpha^1)\frac{1}{2}(z^0 + z^1) \\
&+\frac{1}{2}(k^1\beta^1 + k^1 B^1 k^1 + k^1\Gamma^1 z^1 + k^1\Gamma^1 z^0 \\
&-k^0\beta^0 - k^0 B^0 k^0 - k^0\Gamma^0 z^1 - k^0\Gamma^0 z^0) + \frac{1}{2}(k^1\beta^1 - k^0\beta^0) \qquad \text{(B.51)}
\end{aligned}
$$

Adding and subtracting $\frac{1}{2}k^0\beta^1$, $\frac{1}{2}k^1\beta^0$, $\frac{1}{2}k^0 B^1 k^1$, and $\frac{1}{2}k^0 B^0 k^1$ to the right-hand side of (B.51) yield (3.21).

Proof of Lemma (3.2) (Quadratic Approximation of Value Differences between Two Arbitrary Functions with Different Parameters or Functional Forms). The proof of lemma (3.2) follows the proof of Lemma (2.2).

Proof of Corollary (3.3) (Accounting for the Sum of Value Differences between Two Quadratic Functions with Different "Zero-order" and "First-order" Parameters) (Caves, Christensen, and Diewert's, 1982, pp.1412-1413) *Translog Identity*).

Let the quadratic function $h_Q^t(z, k^t)$ be defined by (3.12). By the Quadratic Identity (Corollary (2.2),

$$
\begin{aligned}
h_Q^t(z^1, k^t) - h_Q^t(z^0, k^t) &= \frac{1}{2}[\nabla_z h_Q^t(z^0, k^t) + \nabla_z h_Q^t(z^1, k^t)] \\
&\quad \cdot(z^1 - z^0) \qquad \text{(B.52)}
\end{aligned}
$$

and, therefore, setting $\lambda = 1/2$, for $t = 0$ and $t = 1$, the first line of the right-hand side of equation (3.25) becomes

$$\begin{aligned} & \frac{1}{2}[h_Q^0(z^1, k^0) - h_Q^0(z^0, k^0)] + \frac{1}{2}[h_Q^1(z^1, k^1) - h_Q^1(z^0, k^1)] \\ = & \frac{1}{2}\left[\nabla_z h_Q^0(z^0, k^0) + \nabla_z h_Q^0(z^1, k^0)\right] \ (z^1 - z^0) \\ & + \frac{1}{2}\left[\nabla_z h_Q^1(z^0, k^1) + \nabla_z h_Q^1(z^1, k^1)\right] \ (z^1 - z^0) \end{aligned} \tag{B.53}$$

Since

$$\nabla_z h_Q^0(z^0, k^0) = a^0 + z^0 A \tag{B.54}$$

$$\nabla_z h_Q^0(z^1, k^0) = a^0 + z^1 A \tag{B.55}$$

$$\nabla_z h_Q^1(z^0, k^1) = a^1 + z^0 A \tag{B.56}$$

$$\nabla_z h_Q^1(z^1, k^1) = a^1 + z^1 A \tag{B.57}$$

where $a^t = \alpha^t + k^t \Gamma^t$, for $t = 0, 1$,

$$\begin{aligned} & \nabla_z h_Q^0(z^1, k^0) + \nabla_z h_Q^1(z^0, k^1) \\ = & (a^0 + z^1 A) + (a^1 + z^0 A) \\ = & (a^0 + z^0 A) + (a^1 + z^1 A) \\ = & \nabla_z h_Q^0(z^0, k^0) + \nabla_z h_Q^1(z^1, k^1) \end{aligned} \tag{B.58}$$

Using (B.58), equation (B.53) becomes

$$\begin{aligned} & \frac{1}{2}[h_Q^0(z^1, k^0) - h_Q^0(z^0, k^0)] + \frac{1}{2}[h_Q^1(z^1, k^1) - h_Q^1(z^0, k^1)] \\ = & \frac{1}{2}\left[\nabla_z h_Q^0(z^0, k^0) + \nabla_z h_Q^1(z^1, k^1)\right] (z^1 - z^0) \end{aligned} \tag{B.59}$$

which is equation (3.27).

Proof of Corollary (3.4) (Accounting for the Sum of Value Differences between Two Quadratic Functions with Different Parameters) (Caves, Christensen, and Diewert, 1982, pp.1412-1413).

By the Quadratic Identity,

$$\begin{aligned} & \left[f_Q^0(z^1) - f_Q^0(z^0)\right] + \left[f_Q^1(z^1) - f_Q^1(z^0)\right] \\ = & \frac{1}{2}\left[\nabla_z f_Q^0(z^0) + \nabla_z f_Q^0(z^1)\right] (z^1 - z^0) \\ & + \frac{1}{2}\left[\nabla_z f_Q^1(z^0) + \nabla_z f_Q^1(z^1)\right] (z^1 - z^0) \end{aligned} \tag{B.60}$$

Adding and subtracting $\frac{1}{2}\left[\nabla_z f_Q^0(z^0) + \nabla_z f_Q^1(z^1)\right](z^1 - z^0)$ to the foregoing equation and taking into account the definition of f_Q^t yield

$$\begin{aligned}
& \left[f_Q^0(z^1) - f_Q^0(z^0)\right] + \left[f_Q^1(z^1) - f_Q^1(z^0)\right] \\
= & \left[\nabla_z f_Q^0(z^0) + \nabla_z f_Q^0(z^1)\right](z^1 - z^0) \\
& + \frac{1}{2}(a^0 + z^1 A^0 + a^1 + z^0 A^1 - a^0 - z^0 A^0 - a^1 - z^1 A^1)(z^1 - z^0) \\
= & \left[\nabla_z f_Q^0(z^0) + \nabla_z f_Q^0(z^1)\right](z^1 - z^0) \\
& + \frac{1}{2}(z^1\, A^0 z^1 + z^0\, A^1 z^1 - z^0\, A^0 z^1 - z^1\, A^1 z^1 \\
& - z^1\, A^0 z^0 - z^0\, A^1 z^0 + z^0\, A^0 z^0 + z^1\, A^1 z^0) \\
= & \left[\nabla_z f_Q^0(z^0) + \nabla_z f_Q^0(z^1)\right](z^1 - z^0) \\
& - \frac{1}{2}(z^1 - z^0)(A^1 - A^0)(z^1 - z^0) \qquad \text{(B.61)}
\end{aligned}$$

or

$$\begin{aligned}
& \left[h_Q^0(z^1, k^0) - h_Q^0(z^0, k^0)\right] + \left[h_Q^1(z^1, k^1) - h_Q^1(z^0, k^1)\right] \\
= & \left[\nabla_z h_Q^0(z^0, k^0) + \nabla_z h_Q^1(z^1, k^1)\right](z^1 - z^0) \\
& - \frac{1}{2}(z^1 - z^0)(A^1 - A^0)(z^1 - z^0) \qquad \text{(B.62)}
\end{aligned}$$

which is equation (3.29).

Proof of Corollary (3.5)

By definition,

$$\begin{aligned}
& \left[f_Q^1(z^0) - f_Q^0(z^0)\right] + \left[f_Q^1(z^1) - f_Q^0(z^1)\right] \\
= & [a_0^1 + a^1\, z^0 \ + \frac{1}{2} z^0\, A^1 z^0 - a_0^0 - a^0\, z^0 \ - \frac{1}{2} z^0\, A^0 z^0] \\
& + [a_0^1 + a^1\, z^1 \ + \frac{1}{2} z^1\, A^1 z^1 - a_0^0 - a^0\, z^1 \ - \frac{1}{2} z^1\, A^0 z^1] \qquad \text{(B.63)}
\end{aligned}$$

Adding and subtracting $\frac{1}{2} z^0\, A^0 z^1$ and $\frac{1}{2} z^0\, A^1 z^1$ and noting that the symmetry of A^t (*that is* $A^t = A^{tT}$) implies $z^0\, A^t z^1 = z^1\, A^t z^0$, the foregoing equation becomes

$$\begin{aligned}
& \left[f_Q^1(z^0) - f_Q^0(z^0)\right] + \left[f_Q^1(z^1) - f_Q^0(z^1)\right] \\
= & 2(a_0^1 - a_0^0) - (a_0 - a_0)(z^0 + z^1) + z^0(A^1 - A^0) z^1 \\
& + \frac{1}{2}(z^1 - z^0)(A^1 - A^0)(z^1 - z^0) \qquad \text{(B.64)}
\end{aligned}$$

which corresponds to (3.30).

The derivation of equations (3.32) and (3.33) follows the footsteps of that of equations (3.29), (3.30), and (3.21).

Proof of Corollary (3.6). (Accounting for Functional Value Ratios of Two Different Quadratic Homothetic Functions).

The proof follows directly that of Corollary (2.3).

Proof of Lemma (4.1). (Accounting for Value Differences between Two General Quadratic Functions.)

The difference $g[f^1_{GQ}(q^1)] - g[f^0_{GQ}(q^0)]$ and $g[h^1_{GQ}(q^1, x^1)] - g[h^0_{GQ}(q^0, x^0)]$ can be decomposed by applying Lemma (3.1) and Corollary (3.1), respectively, thus obtaining:

$$\begin{aligned}
& g[f^1_{GQ}(q^1)] - g[f^0_{GQ}(q^0)] \\
= \; & g\left\{f^1_{GQ}[Z^{-1}(z^1)]\right\} - g\left\{f^0_{GQ}[Z^{-1}(z^0)]\right\} \\
= \; & f^1_Q(z^1) - f^0_Q(z^0) = \frac{1}{2}[\nabla_z f^0_Q(z^0) + \nabla_z f^1_Q(z^1)](z^1 - z^0) \\
& + Pc \\
= \; & \frac{1}{2}\left\{g'[f^0_{GQ}(q^0)] \cdot [\widehat{Z}'(q^0)]^{-1} \cdot \nabla_q f^0_{GQ}(q^0) + g'[f^1_{GQ}(q^1)] \cdot [\widehat{Z}'(q^1)]^{-1} \cdot \nabla_q f^1_{GQ}(q^1)\right\}^T \\
& \cdot[(Z(q^1) - Z(q^0)] + Pc \qquad \text{(B.65)}
\end{aligned}$$

where

$$\begin{aligned}
PC \text{ component} \quad \equiv \quad & (a^1_0 - a^0_0) + (a^1 - a^0)\frac{1}{2}[Z(q^0) + Z(q^1] \\
& + Z(q^0)(A^1 - A^0)Z(q^0), \qquad \text{(B.66)}
\end{aligned}$$

the $(i,i)^{th}$ element of the diagonal matrix $[\widehat{Z}'(q)]^{-1}$is equal to

$$\frac{dq_i}{dz_i} = \frac{dz^{-1}(z_i)}{dz_i} = 1/z'(q_i) \qquad \text{(B.67)}$$

(with $z'(q_i) \neq 0$, by assumption).

Since $f_Q(z) \equiv g[f^t_{GQ}(q)] = g\left\{f^t_{GQ}[Z^{-1}(z)]\right\}$,

$$\begin{aligned}
\nabla_z f_Q(z) &= g'\left\{f^t_{GQ}[Z^{-1}(z)]\right\} \cdot \widehat{Z}'^{-1}(z) \cdot \nabla_z f^t_{GQ}[Z^{-1}(z)] \\
&\quad \text{where the } (i,i)^{th} \text{ element of the diagonal} \\
&\quad \text{matrix } \widehat{Z}'^{-1}(z) \text{ is given by } \frac{dz^{-1}(z_i)}{dz_i} \\
&= g'[f^t_{GQ}(q)] \cdot \widehat{Z}'^{-1}(z) \cdot \nabla_z f^t_{GQ}(q) \qquad \text{using (4.3)} \\
&= g'[f^t_{GQ}(q)] \cdot [\widehat{Z}'(q)]^{-1} \cdot \nabla_z f^t_{GQ}(q) \quad \text{using (4.9)} \quad \text{(B.68)}
\end{aligned}$$

equation (B65) can be rewritten as

$$\begin{aligned}
& g[f^1_{GQ}(q^1)] - g[f^0_{GQ}(q^0)] \\
= \; & \frac{1}{2}\left\{g'[f^0_{GQ}(q^0)] \cdot [\widehat{Z}'(q^0)]^{-1} \cdot \nabla_q f^0_{GQ}(q^0) + g'[f^1_{GQ}(q^1)] \cdot [\widehat{Z}'(q^1)]^{-1} \cdot \nabla_q f^1_{GQ}(q^1)\right\}^T \\
& \cdot[(Z(q^1) - Z(q^0)] + Pc \qquad \text{(B.69)}
\end{aligned}$$

Proof of Theorem (6.1):

If function g and z in (4.1) are, respectively, logarithmic transformation of functional values and variables, that is $g(y) \equiv \ln y$ and $Z(q) \equiv \ln q$, then (6.1) follows directly from (4.9).

Proof of Theorem (6.2).

Let us apply the decomposition procedure (4.9) to the quadratic Box-Cox function (5.4), thus obtaining:

$$\begin{aligned}
[\frac{(f^1_{Q^{r,\lambda}})^\rho - 1}{\rho}] - [\frac{(f^0_{Q^{r,\lambda}})^\rho - 1}{\rho}] &= \frac{1}{2}\sum_{i=1}^{N}[\frac{(q^0_i)^{1-\lambda}}{(f^0_{Q^{\rho,\lambda}})^{1-\rho}} f^0_{Q^{\rho,\lambda},i} \\
&\quad + \frac{(q^1_i)^{1-\lambda}}{(f^1_{Q^{\rho,\lambda}})^{1-r}} f^1_{Q^{\rho,\lambda},i}]\left\{\frac{(q^1_i)^\lambda - 1}{\lambda} - \frac{(q^0_i)^\lambda - 1}{\lambda}\right\} \\
&\quad + \textit{Parameter-change component} \\
&= \frac{1}{2}\sum_{i=1}^{N}[\frac{q^0_i f^0_{Q^{\rho,\lambda},i}}{f^0_{Q^{\rho,\lambda}}}(f^0_{Q^{\rho,\lambda}})^\rho \frac{1}{(q^0)^\lambda} + \\
&\quad + \frac{q^1_i f^1_{Q^{\rho,\lambda},i}}{f^1_{Q^{\rho,\lambda}}}(f^1_{Q^{\rho,\lambda}})^\rho \frac{1}{(q^1)^\lambda}][\frac{(q^1_i)^\lambda}{\lambda} - \frac{(q^0_i)^\lambda}{\lambda}] \\
&\quad + \textit{Parameter-change component} \qquad \text{(B.70)}
\end{aligned}$$

where $f^t_{Q^{\rho,\lambda}} \equiv f^t_{Q^{\rho,\lambda}}(q^t)$, $f^t_{Q^{\rho,\lambda},i} \equiv \partial f^t_{Q^{\rho,\lambda}}/\partial q^t_i$.

By defining $s^t_{Q^{\rho,\lambda},i} \equiv \frac{q^t_i f^t_{Q^{\rho,\lambda},i}}{f^t_{Q^{\rho,\lambda}}}$ with $t = 0, 1$, multiplying equation (B.70) by ρ and rearranging terms we obtain

$$\begin{aligned}(f^1_{Q^{\rho,\lambda}})^\rho - (f^0_{Q^{\rho,\lambda}})^\rho &= \frac{1}{2}\frac{\rho}{\lambda}\sum_{i=1}^{N}\left\{s^0_{Q^{\rho,\lambda}i}\frac{(f^0_{Q^{\rho,\lambda}})^\rho}{(q^0_i)^\lambda} + s^1_{Q^{\rho,\lambda}i}\frac{(f^1_{Q^{\rho,\lambda}})^\rho}{(q^1_i)^\lambda}\right\}[(q^1_i)^\lambda - (q^0_i)^\lambda] \\ &\quad + \textit{Parameter-change component} \cdot \frac{\rho}{\lambda} \qquad \text{(B.71)}\end{aligned}$$

which is equation (6.5).

Proof of Theorem (6.3)

Using the definitions (5.18) and (5.19), we have

$$[f^t_{Q^{\rho,\lambda}}(q)/\sigma^t]^\rho = [f_{Q^r}(q)]^r \qquad \text{where} \quad \lambda = \frac{r}{2} \qquad \text{(B.72)}$$

The functions $[f^t_{Q^{\rho,\lambda}}(q)/\sigma^t]^\rho$ and $[f_{Q^r}(q)]^r$ are homogenous of degree $2\lambda = r$.

Using equation (B.70), and denoting $f^t_{Q^{\rho,\lambda}}(q^t)$ with $f^t_{Q^{\rho,\lambda}}$ to simplify notation, it is straightforward to obtain

$$\begin{aligned}\frac{(f^1_{Q^{\rho,\lambda}}/\sigma^1)^\rho}{(f^0_{Q^{\rho,\lambda}}/\sigma^0)^\rho} - 1 &= \frac{\rho}{r}\sum_{i=1}^{N}\left\{\frac{q^0_i f^0_{Q^{\rho,\lambda}i}/\sigma^0}{f^0_{Q^{\rho,\lambda}}/\sigma^0}\frac{1}{(q^0_i)^{\frac{r}{2}}} + \frac{q^1_i f^1_{Q^{\rho,\lambda}i}/\sigma^1}{f^1_{Q^{\rho,\lambda}}/\sigma^1}\frac{(f^1_{Q^{\rho,\lambda}}/\sigma^1)^\rho}{(f^0_{Q^{\rho,\lambda}}/\sigma^0)^\rho}\frac{1}{(q^1_i)^{\frac{r}{2}}}\right\} \cdot (q^1_i)^{\frac{r}{2}} - (q^0_i)^{\frac{r}{2}} \\ (\text{where } f^t_{Q^{\rho,\lambda}i} &\equiv \partial f^t_{Q^{\rho,\lambda}}/\partial q^t_i \quad \text{and} \quad r = 2\lambda) \\ &= \sum_{i=1}^{N} s^0_{Q^{\rho,\lambda}}\frac{(q^1_i)^{\frac{r}{2}}}{(q^0_i)^{\frac{r}{2}}} - \sum_{i=1}^{N} s^0_{Q^{\rho,\lambda}} + \frac{(f^1_{Q^{\rho,\lambda}}/\sigma^1)^\rho}{(f^0_{Q^{\rho,\lambda}}/\sigma^0)^\rho}\sum_{i=1}^{N} s^1_{Q^{\rho,\lambda}} - \frac{(f^1_{Q^{\rho,\lambda}}/\sigma^1)^\rho}{(f^0_{Q^{\rho,\lambda}}/\sigma^0)^\rho}\sum_{i=1}^{N} s^1_{Q^{\rho,\lambda}}\frac{(q^0_i)^{\frac{r}{2}}}{(q^1_i)^{\frac{r}{2}}}\end{aligned}$$

where $s^t_{Q^{\rho,\lambda}i} = q^t_i f^t_{Q^{\rho,\lambda}i} / \sum_{i=1}^{N} q^t_i f^t_{Q^{\rho,\lambda}i}$ since, by Euler's theorem, $\frac{\rho}{r} = \frac{f^t_{Q^{\rho,\lambda}}}{\sum_{i=1}^{N} q^t_i f^t_{Q^{\rho,\lambda}i}}$, for $t = 0, 1$.

$$= \sum_{i=1}^{N} s^0_{Q^{\rho,\lambda}i}\frac{(q^1_i)^{\frac{r}{2}}}{(q^0_i)^{\frac{r}{2}}} - 1 + \frac{(f^1_{Q^{\rho,\lambda}}/\sigma^1)^\rho}{(f^0_{Q^{\rho,\lambda}}/\sigma^0)^\rho} - \frac{(f^1_{Q^{\rho,\lambda}}/\sigma^1)^\rho}{(f^0_{Q^{\rho,\lambda}}/\sigma^0)^\rho}\sum_{i=1}^{N} s^1_{Q^{\rho,\lambda}i}\frac{(q^1_i)^{-\frac{r}{2}}}{(q^0_i)^{-\frac{r}{2}}} \qquad \text{(B.73)}$$

By rearranging (B.73) and solving for $f^1_{Q^{\rho,\lambda}}/f^0_{Q^{\rho,\lambda}}$, the following decomposition is obtained

$$\frac{f^1_{Q^{\rho,\lambda}}}{f^0_{Q^{\rho,\lambda}}} = \left[\frac{\sum_{i=1}^{N} s^0_{Q^{\rho,\lambda}i}\frac{(q^1_i)^{\frac{r}{2}}}{(q^0_i)^{\frac{r}{2}}}}{\sum_{i=1}^{N} s^1_{Q^{\rho,\lambda}i}\frac{(q^1_i)^{-\frac{r}{2}}}{(q^0_i)^{-\frac{r}{2}}}}\right]^{\frac{1}{\rho}} \cdot \frac{\sigma^1}{\sigma^0} \qquad \text{(B.74)}$$

Moreover, using the definition (5.19) for f_{Q^r} and denoting $f_{Q^r}(q^t)$ with $f^t_{Q^r}$ to simplify notation, from the accounting equation (B.70) we derive:

$$
\begin{aligned}
\frac{(f^1_{Q^r})^r}{(f^0_{Q^r})^r} - 1 &= \sum_{i=1}^{N} \left\{ \frac{q^0_i f^0_{Q^r i}}{f^0_{Q^r}} \frac{1}{(q^0_i)^{\frac{r}{2}}} + \frac{q^1_i f^1_{Q^r i}}{f^1_{Q^r}} \frac{(f^1_{Q^r})^r}{(f^0_{Q^r})^r} \frac{1}{(q^1_i)^{\frac{r}{2}}} \right\} (q^1_i)^{\frac{r}{2}} - (q^0_i)^{\frac{r}{2}} \\
\text{(where } f^t_{Q^r i} &\equiv \partial f^t_{Q^r} / \partial q^t_i) \\
&= \sum_{i=1}^{N} s^0_{Q^r i} \frac{(q^1_i)^{\frac{r}{2}}}{(q^0_i)^{\frac{r}{2}}} - \sum_{i=1}^{N} s^0_{Q^r i} + \frac{(f^1_{Q^r})^r}{(f^0_{Q^r})^r} \sum_{i=1}^{N} s^1_{Q_r i} - \frac{(f^1_{Q^r})^r}{(f^0_{Q^r})^r} \sum_{i=1}^{N} s^1_{Q^r i} \frac{(q^0_i)^{\frac{r}{2}}}{(q^1_i)^{\frac{r}{2}}} \\
&= \sum_{i=1}^{N} s^0_{Q^{\rho,\lambda} i} \frac{(q^1_i)^{\frac{r}{2}}}{(q^0_i)^{\frac{r}{2}}} - 1 + \frac{(f^1_{Q^r})^r}{(f^0_{Q^r})^r} - \frac{(f^1_{Q^r})^r}{(f^0_{Q^r})^r} \sum_{i=1}^{N} s^1_{Q^{\rho,\lambda} i} \frac{(q^1_i)^{-\frac{r}{2}}}{(q^0_i)^{-\frac{r}{2}}},
\end{aligned}
$$

where $s^t_{Q^{\rho,\lambda} i} = s^t_{Q^r i} = q^t_i f^t_{Q^r i} / f^t_{Q^r} = q^t_i f^t_{Q^{\rho,\lambda} i} / \frac{r}{\rho} f^t_{Q^{\rho,\lambda}} = q^t_i f^t_{Q^{\rho,\lambda} i} / \sum_{i=1}^{N} q^t_i f^t_{Q^{\rho,\lambda} i}$, since $f^t_{Q^r} = \frac{1}{\sigma^t} (f^t_{Q^{\rho,\lambda}})^{\frac{\rho}{r}}$ and $f^t_{Q^r i} = \frac{1}{\sigma^t} \frac{\rho}{r} (f^t_{Q^{\rho,\lambda}})^{\frac{\rho}{r}-1} f^t_{Q^{\rho,\lambda} i})$ and, by Euler's theorem, $\frac{r}{\rho} f^t_{Q^{\rho,\lambda}} = \sum_{i=1}^{N} q^t_i f^t_{Q^{\rho,\lambda} i}$,

$$
= \sum_{i=1}^{N} s^0_{Q^{\rho,\lambda} i} \frac{(q^1_i)^{\frac{r}{2}}}{(q^0_i)^{\frac{r}{2}}} - 1 + \frac{(f^1_{Q^r})^r}{(f^0_{Q^r})^r} - \frac{(f^1_{Q^r})^r}{(f^0_{Q^r})^r} \sum_{i=1}^{N} s^1_{Q^{\rho,\lambda} i} \frac{(q^1_i)^{-\frac{r}{2}}}{(q^0_i)^{-\frac{r}{2}}} \tag{B.75}
$$

since $f^t_{Q^r}$ is homogeneous of degree one in q and, by Euler's theorem, $\sum_{i=1}^{N} s^t_{Q^r i} = 1$. By rearranging (B.75) and solving for $f^1_{Q^r}/f^0_{Q^r}$, the following index is obtained:

$$
\frac{f^1_{Q^r}}{f^0_{Q^r}} = I_q \equiv \left[\frac{\sum_{i=1}^{N} s^0_{Q^{\rho,\lambda} i} \frac{(q^1_i)^{\frac{r}{2}}}{(q^0_i)^{\frac{r}{2}}}}{\sum_{i=1}^{N} s^1_{Q^{\rho,\lambda} i} \frac{(q^1_i)^{-\frac{r}{2}}}{(q^0_i)^{-\frac{r}{2}}}} \right]^{\frac{1}{r}} \tag{B.76}
$$

Let us define $I_T \equiv \sigma^1/\sigma^0$, and, taking account of (B.74),

$$
\begin{aligned}
(f^1_{Q^{\rho,\lambda}} / f^0_{Q^{\rho,\lambda}}) / (I_q \cdot I_T) &= \\
I_Y &\equiv \left[\frac{\sum_{i=1}^{N} s^0_{Q^{\rho,\lambda} i} \frac{(q^1_i)^{\frac{r}{2}}}{(q^0_i)^{\frac{r}{2}}}}{\sum_{i=1}^{N} s^1_{Q^{\rho,\lambda} i} \frac{(q^1_i)^{-\frac{r}{2}}}{(q^0_i)^{-\frac{r}{2}}}} \right]^{\frac{1}{\rho}-\frac{1}{r}}
\end{aligned} \tag{B.77}
$$

Proof of Theorem (6.4)

Following Diewert (1976, p. 140), we note that, if both f and f_{Q^r} are homogeneous of degree one, then by Euler's theorem $f(q^*) = q^*\nabla f(q^*)$ and $f_{Q^r}(q^*) = q^*\nabla f_{Q^r}(q^*)$. Moreover, since the partial derivative functions $\partial f/\partial q_i$ and $\partial f_{Q^r}/\partial q_i$ are homogeneous of degree zero, application of Euler's theorem on homogenous functions yields, for $i = 1, 2, ..., N$,

$$\sum_{j=1}^{N} q_j^* \partial^2 f(q^*)/\partial q_i \partial q_j = 0 = \sum_{j=1}^{N} q_j^* \partial^2 f_{Q^r}(q^*)/\partial q_i \partial q_j \tag{B.78}$$

These results imply that (6.22), (6.23), and (6.24) will be satisfied under the hypotheses adopted here if and only if

$$\partial f_{Q^r}(q^*)/\partial q_i = f_i^* \equiv \partial f(q^*)/\partial q_i, \quad i = 1, 2, ..., N \tag{B.79}$$

$$\partial^2 f_{Q^r}(q^*)/\partial q_i \partial q_j = f_{ij}^* \equiv \partial^2 f(q^*)/\partial q_i \partial q_j, \ \ 1 \leq i < j \leq N. \tag{B.80}$$

Thus, the $N(N+1)/2$ independent parameters α_{ij} (for $1 \leq i \leq j \leq N$) can be chosen so that the $N + N(N-1)/2 = N(N+1)/2$ equations (B.79) and (B.80) are satisfied. Assuming that f is positive over its domain of definition, that $q_i^* > 0$, and $r \neq 0$, the coefficients α_{ij}^*, for $1 \leq i \leq j \leq N$, can be found by solving the following equations

$$\begin{aligned} f_{ij}^* &= \frac{1-r}{f_{Q^r}(q^*)} f_i^* f_j^* + \frac{r}{2}[f_{Q^r}(q^*)]^{(1-r)} \alpha_{ij}^* (q_i^*)^{\frac{r}{2}-1} (q_j^*)^{\frac{r}{2}-1}, \\ 1 &\leq i < j \leq N. \end{aligned} \tag{B.81}$$

thus obtaining

$$\alpha_{ij} = \frac{f_{ij}^* - \frac{1-r}{f^*} f_i^* f_j^*}{\frac{r}{2}[f^*]^{1-r} (q_i^* q_j^*)^{\frac{r}{2}-1}}, \quad 1 \leq i < j \leq N \tag{B.82}$$

Equation (B.81) is equivalent to (B.80) if also (B.79) holds. In this case, taking into account that $\alpha_{ij}^* = \alpha_{ji}^*$ for $i \neq j$, α_{ii}^* is found as the solution to the following equation:

$$\sum_{j=1}^{N} [f_{Q^r}(q^*)]^{(1-r)} \alpha_{ij}^* (q_i^*)^{\frac{r}{2}-1} (q_j^*)^{\frac{r}{2}-1} = f_i^* \tag{B.83}$$

thus obtaining

$$\alpha_{ii} = \frac{f_i^* - (q_i^*)^{\frac{r}{2}-1} \sum_{j \neq i}^{N} [f_{Q^r}(q^*)]^{(1-r)} \alpha_{ij}^* (q_j^*)^{\frac{r}{2}-1}}{[f_{Q^r}(q^*)]^{(1-r)} (q_i^*)^{r-2}} \tag{B.84}$$

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